

7102 Analysis 4: Real Analysis Notes

Based on the 2013 spring lectures by Dr N
Sidorova

The Author has made every effort to copy down all the content on the board during lectures. The Author accepts no responsibility what so ever for mistakes on the notes nor changes to the syllabus for the current year. The Author highly recommends that reader attends all lectures, making their own notes and to use this document as a reference only

Overview of course: three parts (chapters)

① Uniform convergence: sequences $\langle a_n \rangle$, $a_n \rightarrow a$ (numbers); sequence of functions $f_n \rightarrow f$ (pointwise conv. (strong), uniform conv., other types)

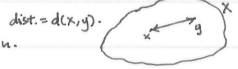
In particular, we will discuss and prove that $\rightarrow \exists$ a continuous but nowhere differentiable function.

Also, we ask if we can approximate any ^{continuous} function f by "nice" functions.

Answer is positive: every continuous function on $[a, b]$ can be approximated by polynomials.

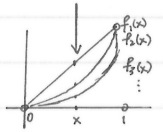
② Fourier series: $\{1, \cos(nx), \sin(nx)\}$. We will extend this set to more general orthogonal systems.

③ Metric spaces: consider a set X s.t. we can define "distance" as an abstract function between two points within.

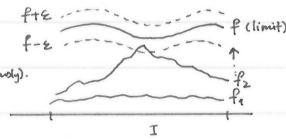


Chapter 1. UNIFORM CONVERGENCE.

Definition let $I \subset \mathbb{R}$, and let $\{f_n\}_{n=1}^{\infty}$ and f be real-valued functions on I . We say that f_n converges to f pointwise if $\forall x \in I, f_n(x) \rightarrow f(x)$. Equivalently, $\forall x \in I, \forall \epsilon > 0 \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, |f_n(x) - f(x)| < \epsilon$. (we have an N for each value of x).
e.g. let $I = (0, 1)$, $f_n(x) = x^n$. then $\forall x \in I, f_n(x) = x^n \rightarrow 0 \Rightarrow f_n$ converges pointwise to the zero function.



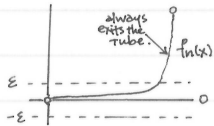
Definition let $I \subset \mathbb{R}$, and let $\{f_n\}_{n=1}^{\infty}$ and f be real-valued functions on I . We say that f_n converges to f uniformly if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, |f_n(x) - f(x)| < \epsilon \forall x \in I$. (we need to find a universal N that would work for all $x \in I$ simultaneously).
Note: We unpack this definition a bit, using a graphical representation. Draw a tube around f , using graphs $f + \epsilon$, $f - \epsilon$. Then if for some $N \in \mathbb{N}$, $n \geq N \Rightarrow f_n$ is in the tube, then it converges uniformly.



Ex We have seen that if $I = (0, 1)$, $f_n(x) = x^n$, then $\{f_n\}$ converges pointwise. Does it converge uniformly?

Ans. The only candidate for a limit, if it converges uniformly, is the same as above (i.e. 0 function)

Draw a tube from $-\epsilon$ to ϵ as shown. Any f_n always exits the tube as $x \rightarrow 1$. i.e. f_n does not converge uniformly to 0 q.e.d.



Theorem 1.1 If $f_n \rightarrow f$ uniformly, then $f_n \rightarrow f$ pointwise.

Corollary If f_n does not converge pointwise, then it does not converge uniformly.

Proof - By contrapositive, q.e.d.

Corollary Suppose $f_n \rightarrow f$ pointwise and $f_n \rightarrow g$ continuously, then $f = g$.

Proof - $f_n \rightarrow g$ continuously $\Rightarrow f_n \rightarrow g$ pointwise. Then $f_n \rightarrow f, g$ pointwise. By uniqueness of limits, $f = g$, q.e.d.

Proof - if $f_n \rightarrow f$ uniformly, then $\forall \epsilon > 0 \exists N$ (depending only on ϵ and not x) s.t. $\forall n \geq N, |f_n(x) - f(x)| < \epsilon \forall x \in I \Rightarrow f_n \rightarrow f$ pointwise, q.e.d.

Testing for pointwise and uniform convergence

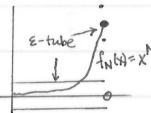
- Test pointwise convergence \rightarrow if there is no pointwise convergence, then there is no uniform convergence.
- If $f_n \rightarrow f$ pointwise, test whether $f_n \rightarrow f$ uniformly.

We now move on to evaluate some examples for practice.

Ex Determine whether the following functions converge pointwise? uniformly?

① $I = [0, 1], f_n(x) = x^n$

Ans. if $x = 1, f_n(x) \rightarrow 1$. if $x = 0, f_n(x) \rightarrow 0$. if $0 < x < 1, f_n(x) \rightarrow 0$. then f_n converges pointwise to f s.t. $f(x) = \begin{cases} 0, & x \in [0, 1) \\ 1, & x = 1 \end{cases}$



We then must determine if $f_n \rightarrow f$ uniformly? We claim it does not, and to do that we apply the negated definition of

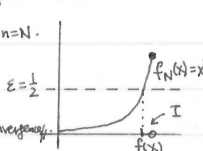
uniform convergence: f_n does not converge to f uniformly, i.e. $\exists \epsilon > 0$ s.t. $\forall N \in \mathbb{N} \exists n \geq N \exists x \in I, |f_n(x) - f(x)| \geq \epsilon$

i.e. we must find an ϵ -tube that is "bad enough". Choose $\epsilon = \frac{1}{2}$, let $N \in \mathbb{N}$ be fixed.

Choose $n = N$, then $f_n(x) = f_N(x) = x^N$. We see that x is any value where the function exits the $\frac{1}{2}$ -tube.

Take left limit, $x^N = \frac{1}{2} \Rightarrow x = \frac{1}{2^{1/N}}$. Then for this $x, |f_n(x) - f(x)| = |f_N(x) - f(x)| = |\frac{1}{2} - 0| = \frac{1}{2} = \epsilon \geq \epsilon \Rightarrow$ no uniform convergence.

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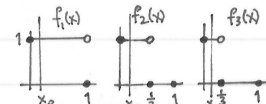


② $I = [0, 1], f_n(x) = \begin{cases} 1 & \text{for } x \in [0, \frac{1}{n}] \\ 0 & \text{for } x \in [\frac{1}{n}, 1] \end{cases}$

soln. if $x = 0, f_n(x) = 1 \rightarrow 1$. if $x = 1, f_n(x) = 0 \rightarrow 0$. For $x \in (0, 1), f_n(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{n}] \\ 0 & \text{if } x \in [\frac{1}{n}, 1] \end{cases} \rightarrow 0$ as $n \rightarrow \infty$. Thus:

$f_n \rightarrow f$ pointwise, where $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in (0, 1] \end{cases}$. Does $f_n \rightarrow f$ uniformly? No. Choose $\epsilon = \frac{1}{2}, \forall N \in \mathbb{N}$. Choose $n = N$,

and pick x s.t. $|f_n(x) - f(x)| = |f_N(x) - f(x)| \geq \frac{1}{2}$. Take $x = \frac{1}{2N}$. then $|f_n(x) - f(x)| = 1 - 0 = 1 \geq \frac{1}{2} = \epsilon$, q.e.d.



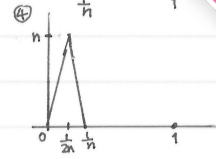
$f_1(x) = f_2(x) = \dots = 1$, but beyond a point goes to 0.

③ $I = [0,1], f_n(x) = \begin{cases} 0 & x \in [\frac{1}{n}, 1] \\ n-n^2x & x \in [0, \frac{1}{n}] \end{cases}$

Soln. Take $x=0$. Then $f_n(0) = n \rightarrow \infty$ (diverges) \Rightarrow no pointwise convergence, \Rightarrow no uniform convergence.

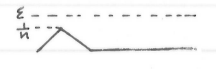
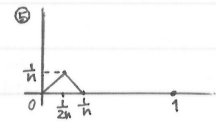
④ $I = [0,1], f_n(x) = \begin{cases} x & x \in [0, \frac{1}{2n}] \\ 2n(x - \frac{1}{2n}) & x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0 & x \in [\frac{1}{n}, 1] \end{cases}$ (see graph marked ④)

Soln. Take $x=0, f_n(0) = 0 \rightarrow 0$; $x=1, f_n(1) = 0 \rightarrow 0$. Then we take $x \in (0,1)$. Consider, for instance, x between $\frac{1}{2n}$ and $\frac{1}{n}$. Then $f_1(x) < f_2(x) < f_3(x)$, but $f_3(x) > f_4(x) > \dots \rightarrow 0$. Hence, for any fixed point $x \in (0,1)$, $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$. Thus, $f_n(x) \rightarrow f(x) = 0$; $f_n \rightarrow f$ pointwise; $f_n \not\rightarrow f$ uniformly. Choose $\epsilon = 100$. Let $N \in \mathbb{N}$, take $n = 101 + N$ and find x s.t. $|f_n(x) - f(x)| \geq 100$. Pick $x = \frac{1}{2n}$. Then $|f_n(x) - f(x)| = |n - 0| = n = 101 + N \geq 100 = \epsilon$



⑤ $I = [0,1], f_n(x) = \begin{cases} x & x \in [0, \frac{1}{n}] \\ 0 & x \in [\frac{1}{n}, 1] \end{cases}$

Soln. As in Ex ④, $f_n \rightarrow f(x) = 0$ pointwise. However, we claim here that $f_n(x) \rightarrow f(x) = 0$ uniformly. Working with definitions, we seek N s.t. given $\epsilon > 0, \exists N$ s.t. $\forall n \geq N \forall x \in I$ s.t. $|f_n(x) - f(x)| < \epsilon$. $\forall n \geq N, \forall x, |f_n(x) - f(x)| = |f_n(x) - 0| = |f_n(x)| < \epsilon$. We want $\frac{1}{n} < \epsilon$, so set $\frac{1}{N} < \epsilon$. $|f_n(x) - f(x)| = |f_n(x)| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$, q.e.d.



⑥ $I = (0, \infty), f_n(x) = x + \frac{1}{x+n}$

Soln. As $n \rightarrow \infty, f_n(x) \rightarrow x + 0 = x \forall x$. Hence pointwise, $f_n \rightarrow f(x) = x$. We then test uniform convergence. Evaluate $|f_n(x) - f(x)|$. $|f_n(x) - f(x)| = |\frac{1}{x+n}| = \frac{1}{x+n} \leq \frac{1}{n}$ (since $x > 0$). Is $\frac{1}{n} < \epsilon$? We suspect that convergence is uniform. $\forall \epsilon > 0$, choose N s.t. $\frac{1}{N} < \epsilon$; let us say, $N = \lfloor \frac{1}{\epsilon} \rfloor + 2$. Then $\forall n \geq N, |f_n(x) - f(x)| = \frac{1}{x+n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$, q.e.d.

Theorem 1.2 (Uniform convergence preserves continuity)

Let $I \subset \mathbb{R}$, let $\{f_n\}_{n=1}^{\infty}$ be real-valued functions on I . Suppose (i) all f_n are continuous and (ii) $f_n \rightarrow f$ uniformly.

Then f is continuous.

Note: Not true for pointwise convergence! e.g. $f_n(x) = x^n$ on $[0,1]$. f_n converges pointwise to discontinuous $f = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$.

Proof:

Let $x \in I$ and prove that f is continuous at x , i.e. $\forall \epsilon > 0 \exists \delta > 0$ s.t. $y \in I, |y-x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be given.

• We know $f_n \rightarrow f$ uniformly $\Rightarrow \exists N$ s.t. $\forall n \geq N, \forall z \in I, |f_n(z) - f(z)| < \frac{\epsilon}{3}$. In particular, $\forall z \in I, |f_n(z) - f(z)| < \frac{\epsilon}{3}$.

• By continuity of f_n at $x, \exists \delta > 0$ s.t. $y \in I, |y-x| < \delta \Rightarrow |f_n(y) - f_n(x)| < \frac{\epsilon}{3}$.

If $y \in I$ and $|y-x| < \delta$, then $|f(y) - f(x)| = |f(y) - f_n(y) + f_n(y) - f_n(x) + f_n(x) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$

From our observations above, and since $x, y \in I$, then $|f(y) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \Rightarrow f$ is continuous, q.e.d.

Covers and subcovers.

Consider the open interval: (a,b) . Let us examine a collection of open intervals. These can be denoted by $\{I_i\}_{i \in A}$, where A is an index set. This reflects the diversity:

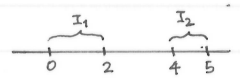
- (1) finite: I_1, I_2, \dots, I_m ; in which case we write $\{I_i\}_{i=1,2,\dots,m}$
- (2) infinite: I_1, I_2, \dots ; in which case we index with natural numbers $\{I_i\}_{i \in \mathbb{N}}$
- (3) infinite: $\{x-1, x+1\}$ $x \in \mathbb{R}$ for instance.

Definition A collection $\{I_i\}_{i \in A}$ of open intervals is a cover of a set $S \subset \mathbb{R}$ if $S \subset \bigcup_{i \in A} I_i$.

Let S be a set and $\{I_i\}_{i \in A}$ be a cover of S . A subcollection of $\{I_i\}_{i \in A}$ is called a subcover of S if this smaller collection is itself a cover of S .

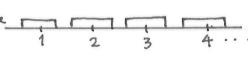
Ex 1 Let $I_1 = (0,2)$ and $I_2 = (4,5)$. Consider the sets $S =$

- (a) $(0,1)$, (b) $[0, \frac{1}{2}]$, (c) $\{1\} \cup (4,4.5)$. Which of these are covers of S ?



Soln. (a) and (b), $\{I_1, I_2\}$ is a cover of $S = (0,1)$ and $S = \{1\} \cup (4,4.5)$. However, $0 \in S$ but $0 \notin \{I_1, I_2\}$.

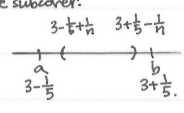
② Let $I_n = (n - \frac{1}{3}, n + \frac{1}{3})$, $n \in \mathbb{N}$. Is $\{I_n\}_{n \in \mathbb{N}}$ a cover for the sets $\mathbb{N}, \mathbb{Z}, \{1\} \cup (3 - \frac{1}{5}, 3 + \frac{1}{5}), (3 - \frac{1}{5}, 3 + \frac{1}{5})$? If so, does it have a finite subcover?



Soln.	set, S :	Is $\{I_n\}_{n \in \mathbb{N}}$ a cover for S ?	Does the cover $\{I_n\}_{n \in \mathbb{N}}$ have a finite subcover?
	\mathbb{N}	yes	no.
	\mathbb{Z}	no	-
	$\{1\} \cup (3 - \frac{1}{5}, 3 + \frac{1}{5})$	yes	yes, we can take $\{I_1, I_3\}$
	$(3 - \frac{1}{5}, 3 + \frac{1}{5})$	yes	yes, we can take I_3 .

③ Consider the set $S = (3 - \frac{1}{5}, 3 + \frac{1}{5})$. In example ②, we had a cover of S which has a finite subcover. Construct a cover of S which has no finite subcover.

Soln. Take $I_n = (3 - \frac{1}{5} + \frac{1}{n}, 3 + \frac{1}{5} - \frac{1}{n})$. Then $\bigcup I_n = (3 - \frac{1}{5}, 3 + \frac{1}{5})$ is a cover of S but I_n is a cover of $(3 - \frac{1}{5}, 3 + \frac{1}{5})$.

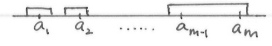


Then, this cover of $(3 - \frac{1}{5}, 3 + \frac{1}{5})$ has no finite subcover.

Question: Are there sets for which every cover has a finite subcover?

Definition A set $K \subseteq \mathbb{R}$ is **compact in \mathbb{R}** if every cover of K (by open intervals) has a finite subcover.

e.g. $S = \{a_1, \dots, a_m\}$ up to finite m is compact.



Theorem 13 (Heine-Borel)

Every closed interval $[a, b]$ is compact in \mathbb{R} .

Proof - NIP: Every cover of $[a, b]$ has a finite subcover. Let $\{I_i\}_{i \in A}$ be some cover of $[a, b]$.

Let $B = \{x \in [a, b] : [a, x] \text{ has a finite subcover}\}$. Obviously, we claim $a \in B \Rightarrow B \neq \emptyset \Rightarrow$ has sup.

Denote $c = \sup B$. Suppose $c \neq b \Rightarrow c < b$. \exists interval I_x s.t. $c \in I_x$ since $c = \sup B$,

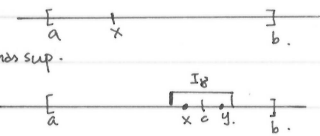
we can pick $x \in B$ and $x \in I_x$ s.t. $x < c$ (by definition) - then by definition of B , the interval

$[a, x]$ can be covered by finite collection I_1, \dots, I_m . Pick $y \in I_x$. Then the interval $[a, y]$ is covered by $I_1, I_2, \dots, I_m, I_x$.

This is a finite collection, so $y \in B$. But $y > c = \sup B \Rightarrow$ contradiction. Thus, $c = b$. We still must establish that $c = b \in B$.

Pick an interval that covers the point b , labelling it newly I_x . Then pick $x \in I_x$, $x < b$. Then $x \in B \Rightarrow$ we need only finitely many intervals to cover $[a, x]$, say I_1, \dots, I_m . Then the interval $[a, b]$ is covered by $I_1, \dots, I_m, I_x \Rightarrow b \in B$. \therefore

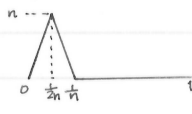
$\{x \in [a, b] : [a, x] \text{ has a finite subcover}\} = [a, b] \Rightarrow [a, b]$ is compact in \mathbb{R} // q.e.d.



Recall that theorem 12 claimed that $\begin{cases} f_n \rightarrow f \text{ uniformly} \\ \text{all } f_n \text{ are continuous} \end{cases} \Rightarrow f \text{ is continuous}$. Can we make a claim about uniform convergence as well, if f is continuous?

No. For instance, recall the example where $f_n \rightarrow 0$ uniformly, shown in graph on right. All f_n are continuous, 0 is continuous; but no uniform convergence

We can however claim this by imposing more constraints.



Theorem 14 (Dini's Theorem)

Let $\{f_n\}_{n=1}^{\infty}, f$ be real-valued functions on $[a, b]$ such that

- (a) $f_n \rightarrow f$ pointwise,
- (b) all f_n are continuous,
- (c) f is continuous, and
- (d) $\forall x \in [a, b], \{f_n(x)\}_{n=1}^{\infty}$ is monotone.

then $f_n \rightarrow f$ uniformly.

Proof - We want $\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \forall x \in [a, b] |f_n(x) - f(x)| < \epsilon$. Let $\epsilon > 0, x \in [a, b]$ fixed.

Since $f_n \rightarrow f$ pointwise, we have $f_n(x) \rightarrow f(x) \Rightarrow \exists N(x) \forall n \geq N(x), |f_n(x) - f(x)| < \frac{\epsilon}{2}$.

In particular, $|f_{N(x)}(x) - f(x)| < \frac{\epsilon}{2}$. Denote $g(y) = f_{N(x)}(y) - f(y)$ (difference of graph from f at $y=x$).

$f_{N(x)}, f$ are continuous at $x \Rightarrow g$ is continuous. Given $\epsilon > 0, \exists \delta(x) > 0$ s.t. $|y-x| < \delta(x), y \in [a, b] \Rightarrow |g(y) - g(x)| < \frac{\epsilon}{2}$.

$\Rightarrow |f_{N(x)}(y) - f(y)| = |g(y)| = |g(y) - g(x) + g(x)| \leq |g(y) - g(x)| + |g(x)| = |g(y) - g(x)| + |f_{N(x)}(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

whenever $y \in [a, b]$ and $|y-x| < \delta(x)$. i.e. there is a neighbourhood of $f_{N(x)}(x)$ in ϵ -tube.

$\forall y \in [a, b], f_n(y)$ is monotone $\Rightarrow f_n(y) - f(y)$ is monotone and converges to 0 $\Rightarrow |f_n(y) - f(y)|$ is decreasing.

$\therefore |f_n(y) - f(y)| \leq |f_{N(x)}(y) - f(y)| < \epsilon$ whenever $y \in [a, b], |y-x| < \delta(x)$.

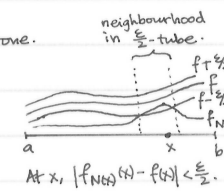
Denote $I(x) = (x - \delta(x), x + \delta(x))$. We have proven that $\forall x \in [a, b], \forall y \in I(x), \forall n \geq N(x) \Rightarrow |f_n(y) - f(y)| < \epsilon$.

In this interval $I(x), f_n \rightarrow f$ uniformly. But depending on x , we have different values for $N(x)$. We want to pick

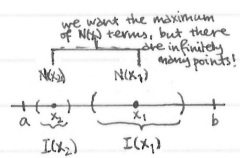
maximum of it, but there are infinitely many points $x \in [a, b]$. We work around this:

Let $\{I(x_i)\}_{i=1}^m, x_i \in [a, b]$ be a cover of $[a, b]$. By Heine-Borel theorem, \exists finite subcover $I(x_1), \dots, I(x_m)$.

Choose $N = \max\{N(x_1), \dots, N(x_m)\}$. Take $n \geq N, x \in [a, b] \Rightarrow x \in I(x_i)$ for some $1 \leq i \leq m, n \geq N \geq N(x_i) \Rightarrow |f_n(x) - f(x)| < \epsilon$ // q.e.d.



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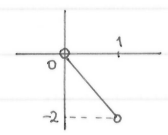
Ex Find an example to show that Dini's Theorem does not work on an open interval.

Soln. Let $(0, 1)$ be the open interval I . Take $f_n(x) = x^n$; certainly, (a) $f_n \rightarrow 0$ pointwise, (b) all $f_n(x) = x^n$ are continuous, (c) $f=0$ is continuous,

(d) $f_n(x)$ is monotone increasing. However, $f_n \not\rightarrow f$ uniformly. // q.e.d.

Definition Let $I \subseteq \mathbb{R}$ and $f: I \rightarrow \mathbb{R}$. Then the **supremum norm** of f is $\|f\|_{\sup} = \sup_{x \in I} |f(x)|$.

e.g. ① $I = \mathbb{R}, f(x) = \sin(x), \|f\|_{\sup} = 1$. ② $I = (0, 1), f(x) = -2x, \|f\|_{\sup} = 2$.



Theorem 1.5 f_n converges to f uniformly on $I \Leftrightarrow \|f_n - f\|_{\text{sup}} \rightarrow 0$ as $n \rightarrow \infty$.

$f_n \rightarrow f$ uniformly, $\exists N \in \mathbb{N}$ s.t. for $n > N$, $\forall x \in I$, $|f_n(x) - f(x)| \leq \epsilon \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n > N, \forall x \in I, \|f_n(x) - f(x)\|_{\text{sup}} \leq \epsilon \Leftrightarrow \|f_n - f\|_{\text{sup}} \rightarrow 0$ q.e.

e.g. ① $I = (0, 1), f_n(x) = x^n$. $f_n \rightarrow 0$ pointwise. $\|f_n - 0\|_{\text{sup}} = \|f_n\|_{\text{sup}} = 1 \not\rightarrow 0 \Rightarrow$ no uniform convergence.

② $I = \mathbb{R}, f_n(x) = \begin{cases} 1 & x \leq n \\ 0 & x > n \end{cases}$. Then $f_n \rightarrow 1$ pointwise. $\|f_n - 1\|_{\text{sup}} = \begin{cases} 0, & x \leq n \\ -1, & x > n \end{cases} \Rightarrow \|f_n - 1\|_{\text{sup}} = 1 \not\rightarrow 0 \Rightarrow$ no uniform convergence.

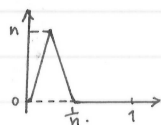
③ $I = \mathbb{R}, f_n(x) = \begin{cases} \frac{1}{n} & x \leq n \\ 0 & x > n \end{cases}$. Then $f_n \rightarrow 0$ pointwise. $\|f_n - 0\|_{\text{sup}} = \|f_n\|_{\text{sup}} = \frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty \Rightarrow$ uniform convergence.

Result that $\left. \begin{matrix} f_n \rightarrow f \text{ uniformly} \\ \text{all } f_n \text{ continuous} \end{matrix} \right\} \Rightarrow f \text{ is continuous}$. We make a similar statement concerning Riemann integrability.

Theorem 1.6 Let $\{f_n\}_{n=1}^{\infty}, f$ be real-valued functions on $[a, b]$, such that

- $f_n \rightarrow f$ uniformly, and
 - all f_n are Riemann integrable,
- then f is Riemann integrable and $\int_a^b f_n(x) dx \rightarrow \int_a^b f(x) dx$.

Note: this theorem does not work if we replace uniform convergence by pointwise convergence. For example, consider the functions f_n on the right. $f_n \rightarrow 0$ pointwise but not uniformly. 0 is integrable, but $\int_0^1 f_n(x) dx = \frac{1}{2} \not\rightarrow 0 = \int_0^1 0 dx$.



Also, sheet 2 Q6 contains an example where f is not integrable.

Result: Let $P \in \mathcal{P}[a, b]$ be a partition of $[a, b]$. $a = t_0 < t_1 < \dots < t_n = b$. Define upper and lower Darboux sums: $U(f, P) = \sum_{i=1}^n \sup_{t_{i-1} \leq x \leq t_i} f(x) (t_i - t_{i-1})$
 $L(f, P) = \sum_{i=1}^n \inf_{t_{i-1} \leq x \leq t_i} f(x) (t_i - t_{i-1})$. We say that f is Riemann integrable iff $\forall \epsilon > 0 \exists P$ s.t. $U(f, P) - L(f, P) < \epsilon$.

Proof - Let $\epsilon > 0$. Since $f_n \rightarrow f$ uniformly, $\exists N$ s.t. $|f_n(x) - f(x)| < \frac{\epsilon}{4(b-a)} \forall x \in [a, b]$. Since f_n is Riemann integrable, there is a partition P s.t.

$$U(f_n, P) - L(f_n, P) < \frac{\epsilon}{4}. \text{ Then } |f_n(x) - f(x)| < \frac{\epsilon}{4(b-a)} \Rightarrow f_n(x) - \frac{\epsilon}{4(b-a)} < f(x) < f_n(x) + \frac{\epsilon}{4(b-a)}, \forall x \in [a, b].$$

$$U(f, P) = \sum_{i=1}^n \sup_{t_{i-1} \leq x \leq t_i} f(x) \cdot (t_i - t_{i-1}) \leq \sum_{i=1}^n \left(\sup_{t_{i-1} \leq x \leq t_i} \left[f_n(x) + \frac{\epsilon}{4(b-a)} \right] \right) (t_i - t_{i-1}) = U(f_n, P) + \frac{\epsilon}{4(b-a)} \sum_{i=1}^n (t_i - t_{i-1}) = U(f_n, P) + \frac{\epsilon}{4}.$$

$$L(f, P) = \sum_{i=1}^n \inf_{t_{i-1} \leq x \leq t_i} f(x) \cdot (t_i - t_{i-1}) \geq L(f_n, P) - \frac{\epsilon}{4}. \text{ Thus, } U(f, P) - L(f, P) \leq U(f_n, P) + \frac{\epsilon}{4} - L(f_n, P) + \frac{\epsilon}{4} = \frac{\epsilon}{2} + [U(f_n, P) - L(f_n, P)] = \frac{3\epsilon}{4} < \epsilon, \text{ q.e.d.}$$

$$\left| \int_a^b f_n(x) dx - \int_a^b f(x) dx \right| = \left| \int_a^b (f_n(x) - f(x)) dx \right| \leq \int_a^b |f_n(x) - f(x)| dx \text{ by triangle inequality of integrals. We estimate by supremum norm: (just a real number)}$$

$$\left| \int_a^b f_n(x) dx - \int_a^b f(x) dx \right| \leq \int_a^b |f_n(x) - f(x)| dx \leq \int_a^b \|f_n(x) - f(x)\|_{\text{sup}} dx = \|f_n(x) - f(x)\|_{\text{sup}} \int_a^b dx = (b-a) \|f_n(x) - f(x)\|_{\text{sup}} \rightarrow 0 \text{ as } f_n \rightarrow f \text{ uniformly}$$

Hence, $\int_a^b f_n(x) dx \rightarrow \int_a^b f(x) dx$ q.e.d.

How about differentiability? Is there a condition similarly for differentiability? No.

e.g. ① $I = \mathbb{R}, f_n(x) = \frac{1}{n} \sin(n^2 x)$. Then $f_n \rightarrow 0$ pointwise and uniformly: $\|f_n - 0\|_{\text{sup}} = \frac{1}{n} \rightarrow 0$. $f_n'(x) = n \cos(n^2 x)$. At $x=0$,

$f_n'(0) = n$, which does not converge. i.e. f_n' does not converge, not even pointwise; although $f=0$ certainly does.

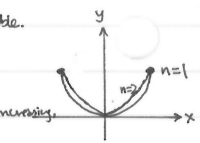
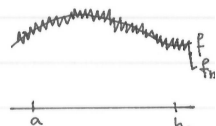
② In the previous example, we have $f_n \rightarrow f$ uniformly, all f_n differentiable. Although f_n' did not converge, at least we had f differentiable.

Now, we construct $f_n \rightarrow f$ uniformly, all f_n differentiable; but f is not differentiable. Take $I = [-1, 1], f_n(x) = |x|^{1+\frac{1}{n}}$.

Clearly, $f_n \rightarrow |x|$ pointwise. Using Dini's Theorem, $f_n \rightarrow |x|$ pointwise, all f_n are continuous, $|x|$ is continuous and f_n' is monotone increasing.

$\Rightarrow f_n \rightarrow f$ uniformly. We see that at $x=0$: $\lim_{h \rightarrow 0} \frac{f_n(h) - f_n(0)}{h} = \lim_{h \rightarrow 0} \frac{|h|^{1+\frac{1}{n}}}{h} = \lim_{h \rightarrow 0} |h|^{\frac{1}{n}} \text{sign}(h) = 0$, so $f_n(x)$ are differentiable,

but $f(x) = |x|$ is not differentiable.



Series

A series is given by $\sum_{n=1}^{\infty} g_n$, where g_n are functions s.t. $g_n: I \rightarrow \mathbb{R}$. Let $f_n(x)$ be the partial sum of the series from 1 to n : $f_n(x) = \sum_{i=1}^n g_i(x)$.

This gives us a sequence of partial sums.

Definition A series $\sum_{n=1}^{\infty} g_n$ converges pointwise on I if f_n converges pointwise on I .

If $f_n \rightarrow f$, then we call f the sum of the series, and write $f = \sum_{n=1}^{\infty} g_n$.

The series $\sum_{n=1}^{\infty} g_n$ converges uniformly on I if f_n converges uniformly on I .

Ex 1 The series $\sum_{n=0}^{\infty} x^n$ converges on $(-1, 1)$, diverges otherwise \Rightarrow converges pointwise on $(-1, 1)$: let $f_n(x) = \sum_{i=0}^n x^i = \frac{1-x^{n+1}}{1-x}$ for $x \in (-1, 1)$.

Does it converge uniformly on $(-1, 1)$? on $[r, r]$ where $0 < r < 1$?

Soln. We know that as $n \rightarrow \infty, f_n(x) \rightarrow \frac{1}{1-x}$ for all $x \in (-1, 1)$. Then $|f_n(x) - f(x)| = \left| \frac{1-x^{n+1}}{1-x} - \frac{1}{1-x} \right| = \frac{|x|^{n+1}}{1-x}$.

on $x \in (-1, 1)$: $\|f_n - f\|_{\text{sup}} = \sup_{x \in (-1, 1)} \frac{|x|^{n+1}}{1-x} = \infty$ (as $x \rightarrow 1$) $\not\rightarrow 0 \Rightarrow$ no uniform convergence on $(-1, 1)$.

on $x \in [r, r]$: $|x|^{n+1} \leq r^{n+1}$ and $1-x \geq 1-r$. then, $\|f_n - f\|_{\text{sup}} = \sup_{x \in [r, r]} \frac{|x|^{n+1}}{1-x} \leq \sup_{x \in [r, r]} \frac{r^{n+1}}{1-r} \rightarrow 0$.

thus, uniform convergence on $[r, r]$.



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Definition Let $I \subset \mathbb{R}$ and $\{f_n\}_{n=1}^{\infty}$ be real-valued functions on I . The sequence $\{f_n\}$ is called a uniform Cauchy sequence if $\forall \epsilon > 0, \exists N \in \mathbb{N} \forall n, m \geq N \|f_n - f_m\|_{\text{sup}} < \epsilon$.

Theorem 1.7 (Central Principle of Uniform Convergence).

A sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $I \iff \{f_n\}_{n=1}^{\infty}$ is a uniform Cauchy sequence on I .

Proof (\implies) : suppose $f_n \rightarrow f$ uniformly on I , i.e. given $\epsilon > 0 \exists N \in \mathbb{N}$ s.t. $\forall n > N, |f_n - f| < \frac{\epsilon}{4} \forall x \in I$. Then we have:

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} \quad \forall n, m \geq N, x \in I.$$

Thus, $\|f_n - f_m\|_{\text{sup}} < \frac{\epsilon}{2} < \epsilon \quad \forall n, m \geq N \implies \{f_n\}_{n=1}^{\infty}$ is a uniform Cauchy sequence, q.e.d.

(\impliedby) : Suppose $\{f_n\}_{n=1}^{\infty}$ is a uniform Cauchy sequence. Then $\forall \epsilon > 0, \exists N \in \mathbb{N} \forall n, m \geq N, \sup_{x \in I} |f_n(x) - f_m(x)| < \frac{\epsilon}{2}$. In particular, for n, m fixed, $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$ for any $x \in I$. Then $\forall x \in I \implies \{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence of numbers. Then $f_n(x)$ converges to some value $f(x)$.

Now, $|f_n(x) - f_m(x)| < \epsilon \quad \forall n, m \geq N$. Let $m \rightarrow \infty$, then $|f_n(x) - f(x)| \leq \lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| \leq \epsilon \quad \forall n \geq N, x \in I \implies f_n \rightarrow f$ uniformly on $x \in I$.
or equal to!

Theorem 1.8 (Weierstrass M-test).

Let $\sum_{n=1}^{\infty} g_n$ be a series of real-valued functions on $I \subset \mathbb{R}$. Suppose there is a sequence of numbers $\{M_n\}_{n=1}^{\infty} \in \mathbb{R}$ s.t.

- $|g_n(x)| \leq M_n \quad \forall n \quad \forall x \in I$, and $\sum_{n=1}^{\infty} M_n$ converges,

then $\sum_{n=1}^{\infty} g_n$ converges uniformly.

Note: This is similar to the comparison-test for series, except that $\forall x \in I, g_n$ is being compared with a fixed constant M_n .

Proof - $f_n(x) = \sum_{i=1}^n g_i(x)$. We want to prove that $\{f_n\}$ is a Cauchy sequence (and thus by central principle of uniform convergence, f_n converges uniformly).

Let $\epsilon > 0$. $\sum_{n=1}^{\infty} M_n$ converges $\iff S_n = \sum_{i=1}^n M_i$ converges $\iff \{S_n\}$ is a Cauchy sequence $\iff \exists N \in \mathbb{N}, \forall n, m \geq N, |S_n - S_m| < \frac{\epsilon}{2}$.

We have, assuming $n > m$ wlog, $|f_n(x) - f_m(x)| = \left| \sum_{i=1}^n g_i(x) - \sum_{i=1}^m g_i(x) \right| = \left| \sum_{i=m+1}^n g_i(x) \right| \leq \sum_{i=m+1}^n |g_i(x)| \leq \sum_{i=m+1}^n M_i = S_n - S_m < \frac{\epsilon}{2} \quad \forall x \in I$.

then $\|f_n - f_m\|_{\text{sup}} < \frac{\epsilon}{2} < \epsilon \quad \forall n, m \geq N \implies \{f_n\}_{n=1}^{\infty}$ is uniform Cauchy sequence \implies by CPUC, $f_n(x)$ converges uniformly, q.e.d.

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Theorem 1.9 If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $g_n \rightarrow 0$ uniformly.

Note: In particular, if $g_n \rightarrow 0$ uniformly, then $\sum_{n=1}^{\infty} g_n$ does not converge uniformly.

Proof - $\sum_{n=1}^{\infty} g_n$ converges uniformly $\implies f_n$ converges uniformly, where $f_n = \sum_{i=1}^n g_i$. By CPUC, f_n is a uniform Cauchy sequence

$$\implies \forall \epsilon > 0 \exists N \forall n, m \geq N, \|f_n - f_m\|_{\text{sup}} < \epsilon. \text{ In particular, } \|f_{n+1} - f_n\|_{\text{sup}} < \epsilon \implies \|g_{n+1}\|_{\text{sup}} < \epsilon \implies g_{n+1} \rightarrow 0 \text{ uniformly, } g_n \rightarrow 0 \text{ uniformly, q.e.d.}$$

Note: This theorem does not work in the opposite direction. For instance, $\sum_{n=1}^{\infty} \frac{x^n}{n}$ does not converge uniformly although $\frac{x^n}{n} \rightarrow 0$ uniformly.

Ex 1 ① Does the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$ converge uniformly?

Soln. $\frac{\sin(nx)}{2^n} \leq \frac{1}{2^n}$. $\sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$, so by Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$ converges uniformly. \parallel

② Does $\sum_{n=1}^{\infty} (\sin x)^n$ on $x \in (0, \frac{\pi}{2})$ converge uniformly?

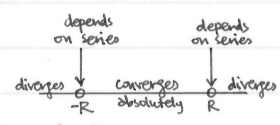
Soln. We check if $(\sin x)^n \rightarrow 0$. $\|(\sin x)^n\|_{\text{sup}} = 1 \not\rightarrow 0 \implies \sum_{n=1}^{\infty} (\sin x)^n$ does not converge uniformly. \parallel

③ Does $\sum_{n=1}^{\infty} \frac{1}{n^2+x}$ converge uniformly on $[0, \infty)$?

Soln. $\frac{1}{n^2+x} \leq \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, so by M-test, $\sum_{n=1}^{\infty} \frac{1}{n^2+x}$ converges uniformly. \parallel

Consider a power series $\sum_{n=0}^{\infty} a_n x^n$, with radius of convergence R . Then, series converges absolutely for $|x| < R$, diverges for $|x| > R$.

Recall that for $\sum_{n=0}^{\infty} x^n$, with $r = 1 - \epsilon$, the series converges uniformly on $[-r, r]$ but does not converge uniformly on $(-1, 1)$.



Theorem 1.10 Let R be the radius of convergence of a power series $\sum_{n=1}^{\infty} a_n x^n$. Let $0 < r < R$. Then the series $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$.

Proof - If $|x| \leq r$, then $|a_n x^n| \leq |a_n| r^n = |a_n r^n| \quad \forall x \in [-r, r]$. Then $\sum_{n=1}^{\infty} |a_n r^n| < \infty$ since the series converges absolutely at the point r .

By the M-test, $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$, q.e.d.

Recall that $\left. \begin{matrix} f_n \rightarrow f \text{ uniformly} \\ \text{all } f_n \text{ are continuous} \end{matrix} \right\} \implies f \text{ is continuous. Some result holds for integrability.}$

Theorem 1.11 If all g_n are continuous on I , and $\sum_{n=1}^{\infty} g_n$ converges uniformly on I , then the function $\sum_{n=1}^{\infty} g_n$ is continuous on I .

Proof - Denote by $f_n = g_1 + \dots + g_n$, the n^{th} partial sum. All g_n are continuous $\implies f_n$ is continuous. f_n converges uniformly.

then $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ is continuous $\implies \sum_{n=1}^{\infty} g_n$ is continuous, q.e.d.

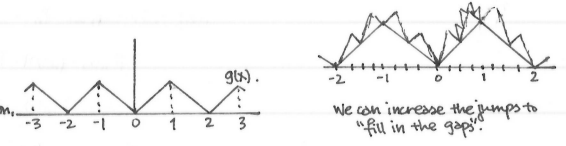
Theorem 1.12 If all g_n are Riemann integrable on $[a,b]$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly on $[a,b]$, then $\sum_{n=1}^{\infty} g_n$ is Riemann integrable on $[a,b]$, and $\int_a^b (\sum_{n=1}^{\infty} g_n(x)) dx = \sum_{n=1}^{\infty} \int_a^b g_n(x) dx$.

Proof - Denote $f_n = g_1 + \dots + g_n$. Since all g_i are Riemann integrable, f_n is Riemann integrable. Since $\sum_{n=1}^{\infty} g_n$ converges uniformly, f_n converges uniformly to f . We claim that $\sum_{n=1}^{\infty} g_n$ is Riemann integrable. Then $\int_a^b (\sum_{n=1}^{\infty} g_n(x)) dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \lim_{n \rightarrow \infty} \int_a^b (g_1 + \dots + g_n) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_a^b g_i(x) dx = \sum_{i=1}^{\infty} \int_a^b g_i(x) dx$ q.e.d.

Theorem 1.13 There is a continuous function on \mathbb{R} which is nowhere differentiable.

Proof - We define $g(x) = |x|$ on $[-1,1]$ and extend it to \mathbb{R} 2-periodically.

Define $f(x) = \sum_{n=1}^{\infty} (\frac{3}{4})^n g(\frac{x}{4^n})$. The 4^n term scales the function in the x-direction.



while the factor of $(\frac{3}{4})^n$ ensures that terms of sequence converge to 0.

Claim 1: The series $\sum_{n=1}^{\infty} (\frac{3}{4})^n g(\frac{x}{4^n})$ converges uniformly on \mathbb{R} . $|(\frac{3}{4})^n g(\frac{x}{4^n})| \leq (\frac{3}{4})^n \forall x \forall n$. Naturally $\sum (\frac{3}{4})^n < \infty$, so series converges uniformly by M-test. $\Rightarrow f$ is well-defined.

Claim 2: f is a continuous function. Since each $(\frac{3}{4})^n g(\frac{x}{4^n})$ is a product of continuous functions, it is continuous. Since series converges uniformly, f is continuous.

Claim 3: f is nowhere differentiable. Fix arbitrary $x \in \mathbb{R}$. $\frac{f(x+h) - f(x)}{h}$ is what we consider. We will construct a sequence $h_m \rightarrow 0$ s.t. $|\frac{f(x+h_m) - f(x)}{h_m}| \rightarrow \infty$.

Choose $h_m = \frac{1}{2} 4^m$ if \exists integer in $(4^m x - \frac{1}{2}, 4^m x)$ left half. Choose $h_m = \frac{1}{2} 4^m$ if \exists integer in $(4^m x, 4^m x + \frac{1}{2})$ right half. (either of them otherwise).

then $\frac{f(x+h_m) - f(x)}{h_m} = \frac{\sum_{n=1}^{\infty} (\frac{3}{4})^n g(\frac{x+h_m}{4^n}) - \sum_{n=1}^{\infty} (\frac{3}{4})^n g(\frac{x}{4^n})}{h_m} = \frac{1}{h_m} \sum_{n=1}^{\infty} (\frac{3}{4})^n [g(\frac{x+h_m}{4^n}) - g(\frac{x}{4^n})]$

Fix m , then let $\frac{(\frac{3}{4})^n g(\frac{x+h_m}{4^n}) - g(\frac{x}{4^n})}{h_m} = a_n$ by definition.

Claim: $|a_n| \leq 3^n$ if $n > m$ (a), $|a_n| \leq 3^m$ if $n = m$ (b), $|a_n| \leq 3^m$ if $n < m$ (c). For (a), $|g(\frac{x+h_m}{4^n}) - g(\frac{x}{4^n})| = |g(4^n x \pm \frac{1}{2} 4^{n-m}) - g(4^n x)|$. Then $n-m \in \mathbb{N}$.

As such, $\frac{1}{2} 4^{n-m}$ is divisible by 2. By periodicity of g with period 2, $g(k+2) = g(k) \Rightarrow |a_n| = 0$

For (b), $|g(\frac{x+h_m}{4^m}) - g(\frac{x}{4^m})| = |g(4^m x \pm \frac{1}{2}) - g(4^m x)| = |g(4^m x \pm \frac{1}{2}) - g(4^m x)| = |\frac{1}{2} 4^m x \pm \frac{1}{2} - 4^m x| = \frac{1}{2}$.

Then $|a_m| = (\frac{3}{4})^m \cdot \frac{1}{2} 4^m = 3^m$. We finally evaluate for (c):

For (c), $|g(\frac{x+h_m}{4^n}) - g(\frac{x}{4^n})| = |g(4^n x \pm \frac{1}{2} 4^{n-m}) - g(4^n x)| \leq |(4^n x \pm \frac{1}{2} 4^{n-m}) - (4^n x)| = \frac{1}{2} 4^{n-m}$.

This is a valid inequality: we want to explain why $|g(a) - g(b)| \leq |a - b|$. This is true if a, b are in same interval, but even more so if not.

$\therefore |a_n| \leq (\frac{3}{4})^n \frac{1}{2} 4^{n-m} = (\frac{3}{4})^n 4^n = 3^n$, and the inequality holds.

Having proven our subsidiary claim, we see that $|\frac{f(x+h_m) - f(x)}{h_m}| = |\sum_{n=1}^{\infty} \frac{(\frac{3}{4})^n g(\frac{x+h_m}{4^n}) - g(\frac{x}{4^n})}{h_m}|$.

then $|\frac{f(x+h_m) - f(x)}{h_m}| = |a_1 + a_2 + \dots + a_{m-1} + a_m| \geq |a_m| - |a_1| - \dots - |a_{m-1}| \geq 3^m - (|a_1| + |a_2| + \dots + |a_{m-1}|) \geq 3^m - (3^1 + 3^2 + \dots + 3^{m-1}) = 3^m - \frac{3^m - 1}{3 - 1} = \frac{3^m + 1}{2} \rightarrow \infty$.

and $|a_m| - |a_1| - \dots - |a_{m-1}| \geq |a_m| - (|a_1| + |a_2| + \dots + |a_{m-1}|) \geq 3^m - (3^1 + 3^2 + \dots + 3^{m-1}) = 3^m - \frac{3^m - 1}{3 - 1} = \frac{3^m + 1}{2} \rightarrow \infty$.

hence, f is nowhere differentiable $\Rightarrow |\frac{f(x+h_m) - f(x)}{h_m}| \rightarrow \infty$, f is not differentiable at arbitrary $x \in \mathbb{R}$ q.e.d.

Note: this theorem was first proved by Weierstrass, who instead used a cosine function rather than $|x|$, i.e. he chose $f(x) = \sum_{n=1}^{\infty} a^n \cos(b^n x)$ rather than $f(x) = \sum_{n=1}^{\infty} (\frac{3}{4})^n g(\frac{x}{4^n})$.

Approximation of Continuous Functions by Polynomials
defined on the closed interval $[a,b]$.

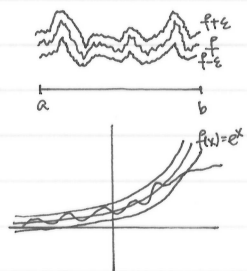
Consider a continuous function f_x . Take $\epsilon > 0$, and construct the epsilon tube from $f - \epsilon$ to $f + \epsilon$.

Can we always find a polynomial lying within the ϵ -tube? Yes, but only on $[a,b]$.

For example, ① take $f(x) = \frac{1}{x}$ on $(0,1)$. then we see that $f(x)$ is unbounded, and near $x=0$, the ϵ -tube also goes to infinity.

however, polynomials cannot converge to infinity near $x=0 \Rightarrow$ not true on $(0,1)$

② Not true on \mathbb{R} : take $f(x) = e^x$. As $x \rightarrow \infty$, polynomial growth cannot match exponential growth.



Definition We define $P_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $n=1,2,3,\dots$, $0 \leq k \leq n$. these polynomials correspond to the binomial distribution $B(n,x)$.

let us consider Y_n to be a binomial random variable. then $Y_n, x \approx nx$ for large n . then $\frac{Y_n}{n} \approx x$, $f(\frac{Y_n}{n}) \approx f(x)$.

To get rid of the randomness, we use the mean/expectation value. $f(x) \approx E[f(\frac{Y_n}{n})] = \sum_{k=0}^n f(\frac{k}{n}) P(Y_n = k) = \sum_{k=0}^n f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}$.

We claim that this polynomial, $B_n^f(x) = \sum_{k=0}^n f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}$, the Bernstein polynomial, gives an approximation to f .

Theorem 1.14 (Weierstrass's Approximation theorem) on $[0,1]$.

let $f: [0,1] \rightarrow \mathbb{R}$ be a continuous function. then $B_n^f \rightarrow f$ uniformly on $[0,1]$.

Lemma let $x \in [0,1]$, then (expectation) (variance)
(a) $\sum_{k=0}^n P_{nk}(x) = 1$, (b) $\sum_{k=0}^n k P_{nk}(x) = nx$, (c) $\sum_{k=0}^n (k-nx)^2 P_{nk}(x) = nx(1-x)$.

Proof - (a)

(c) $k(k-1) \binom{n}{k} = k(k-1) \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(k-2)!(n-k)!} \cdot \frac{(n-1)!}{(n-1)!} = \frac{(n-2)!}{(k-2)!(n-k-2)!} \cdot \frac{(n-1)!}{(n-1)!} = \binom{n-2}{k-2} n(n-1)$. then we have, by definition, $\sum_{k=0}^n k(k-1) P_{nk}(x) = \sum_{k=0}^n k(k-1) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=2}^n \binom{n-2}{k-2} n(n-1) x^k (1-x)^{n-k} = n(n-1) x^2 \sum_{k=2}^n \binom{n-2}{k-2} x^{k-2} (1-x)^{(n-2)-(k-2)} = n(n-1) x^2 \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{(n-2)-k}$

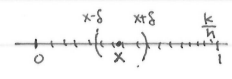
Hence, since $\sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{(n-2)-k} = 1$ from part (a), $\sum_{k=0}^n k(k-1)P_k(x) = n(n-1)x^2$. Then we have
 $\sum_{k=0}^n (k-n)^2 P_k(x) = \sum_{k=0}^n (k^2 - 2knx + n^2 x^2) P_k(x) = \sum_{k=0}^n [k(k-1) + k - 2knx + n^2 x^2] P_k(x) = \sum_{k=0}^n k(k-1)P_k(x) + \sum_{k=0}^n k P_k(x) - 2nx \sum_{k=0}^n k P_k(x) + n^2$
 $= n(n-1)x^2 + nx - 2nx^2 + n^2 x^2 = nx(1-x)$ q.e.d.

With that, we will prove Theorem 1.14:

Proof - MTR: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, |B_n^f(x) - f(x)| < \epsilon$. Let $\epsilon > 0$. f is continuous on $[0,1] \Rightarrow f$ is uniformly continuous on $[0,1] \Rightarrow \exists \delta > 0$ s.t. if $x, y \in [0,1]$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$. Then

$$|B_n^f(x) - f(x)| = \left| \sum_{k=0}^n f\left(\frac{k}{n}\right) P_k(x) - f(x) \sum_{k=0}^n P_k(x) \right| = \left| \sum_{k=0}^n \left(f\left(\frac{k}{n}\right) - f(x) \right) P_k(x) \right| \leq \sum_{k=0}^n \left| f\left(\frac{k}{n}\right) - f(x) \right| P_k(x)$$

We can split this into two sums and evaluate them separately: $\sum_{k: |\frac{k}{n}-x| < \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| P_k(x) + \sum_{k: |\frac{k}{n}-x| \geq \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| P_k(x)$



For the first term, $|\frac{k}{n} - x| < \delta \Rightarrow \left| f\left(\frac{k}{n}\right) - f(x) \right| < \frac{\epsilon}{2}$ by uniform continuity. For the second sum,

$$\left| f\left(\frac{k}{n}\right) - f(x) \right| \leq \left| f\left(\frac{k}{n}\right) \right| + |f(x)| \leq 2 \|f\|_{\text{sup}}. \text{ Hence, } |B_n^f(x) - f(x)| < \frac{\epsilon}{2} \sum_{k: |\frac{k}{n}-x| < \delta} P_k(x) + 2 \|f\|_{\text{sup}} \sum_{k: |\frac{k}{n}-x| \geq \delta} P_k(x).$$

Surely, $\sum_{k: |\frac{k}{n}-x| < \delta} P_k(x) \leq \sum_{k=0}^n P_k(x) = 1$, so $\frac{\epsilon}{2} \sum_{k: |\frac{k}{n}-x| < \delta} P_k(x) \leq \frac{\epsilon}{2}$.

$$\text{We see also that } |\frac{k}{n} - x| \geq \delta \Rightarrow \frac{(k-nx)^2}{n^2} \geq \delta^2 \Rightarrow \frac{(k-nx)^2}{n^2 \delta^2} \geq 1. \text{ Then } |B_n^f(x) - f(x)| < \frac{\epsilon}{2} + 2 \|f\|_{\text{sup}} \sum_{k: |\frac{k}{n}-x| \geq \delta} P_k(x)$$

$$\text{i.e. } |B_n^f(x) - f(x)| < \frac{\epsilon}{2} + 2 \|f\|_{\text{sup}} \sum_{k=0}^n \frac{(k-nx)^2}{n^2 \delta^2} P_k(x) = \frac{\epsilon}{2} + 2 \|f\|_{\text{sup}} \cdot \frac{1}{n^2 \delta^2} nx(1-x) \leq \frac{\epsilon}{2} + \frac{2 \|f\|_{\text{sup}}}{n \delta^2}$$

choose N s.t. $\frac{2 \|f\|_{\text{sup}}}{N \delta^2} < \frac{\epsilon}{2}$ (density possible $\because f$ is bounded). Then for any $n \geq N, |B_n^f(x) - f(x)| < \frac{\epsilon}{2} + \frac{2 \|f\|_{\text{sup}}}{n \delta^2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ q.e.d.

We can expand this theorem to accommodate the general case for an arbitrary interval.

Theorem 1.15 (Weierstrass's Approximation Theorem on $[a,b]$)

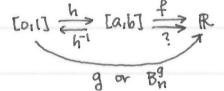
Let $f: [a,b] \rightarrow \mathbb{R}$ be a continuous function. Then \exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ s.t. $P_n \rightarrow f$ uniformly on $[a,b]$.

Proof - Use the above theorem as a base. We find a mapping g s.t. $[0,1] \xrightarrow{h} [a,b] \xrightarrow{f} \mathbb{R}$. Let h be the linear function $h(t) = a + (b-a)t$.

then we define $g: [0,1] \rightarrow \mathbb{R}, g(t) = f(h(t)) = f(x(t))$, which is continuous on $[0,1]$ as a composition of continuous functions.

Define $P_n(x) = B_n^g(h^{-1}(x)) = B_n^g\left(\frac{x-a}{b-a}\right)$. We claim that $P_n: [a,b] \rightarrow \mathbb{R}$ converges to f uniformly.

$$\|P_n - f\|_{\text{sup}} = \sup_{x \in [a,b]} |P_n(x) - f(x)| = \sup_{x \in [a,b]} \left| B_n^g\left(\frac{x-a}{b-a}\right) - f(x) \right| = \sup_{t \in [0,1]} |B_n^g(t) - g(t)| = \|B_n^g - g\|_{\text{sup}} \rightarrow 0.$$



Chapter 2
FOURIER SERIES.

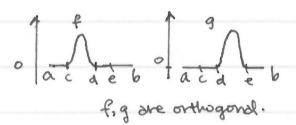
consider the space $R[a,b]$, which denotes $R[a,b] = \{f: [a,b] \rightarrow \mathbb{R} \text{ s.t. } f \text{ is Riemann integrable}\}$.

We also write $\langle f, g \rangle = \int_a^b f(x)g(x) dx$, the inner product of $f, g \in R[a,b]$. Then $\langle \cdot, \cdot \rangle$ satisfies all properties of an inner product from MATH2201,

except positivity. $\exists f \neq 0$ but $\langle f, f \rangle = 0$, since f need not be continuous e.g. $f(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus \{0\} \\ 1, & x = 0 \end{cases}$.

Definition Two functions $f, g \in R[a,b]$ are **orthogonal** if $\langle f, g \rangle = \int_a^b f(x)g(x) dx = 0$.

e.g. see the graphs of the two functions on right, which are orthogonal.



Definition A sequence of Riemann-integrable functions $\{f_n\}_{n=1}^{\infty}$ is an **orthogonal system** if $\langle f_n, f_m \rangle = 0 \forall n \neq m$.

It is an **orthonormal system** if additionally, $\langle f_n, f_n \rangle = 1 \forall n$.

We call, for $f \in R[a,b]$, $\|f\|_2 = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b f(x)^2 dx}$ the **norm** of f .

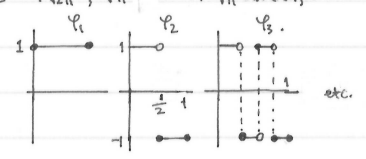
Examples of systems:

① Trigonometric orthogonal system on $[-\pi, \pi]$: $\{1, \cos(nx), \sin(nx); n \in \mathbb{N}\}$. orthonormal system is $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx) \right\}$.

② on $[0,1]$ - the sequence of functions as shown on the right.

Then $\{f_n\}$ is an orthogonal system: $\langle f_n, f_n \rangle = \int_0^1 f_n^2(x) dx = \int_0^1 1 dx = 1$.

$\langle f_m, f_n \rangle = 0$ if $n \neq m$: we try $\langle f_2, f_3 \rangle = \int_0^1 f_2(x)f_3(x) dx = 0$ by drawing a picture of $f_2(x)f_3(x)$.



Definition Let $f \in R[a,b]$ and let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal system on $[a,b]$. Let $a_n = \langle f, f_n \rangle = \int_a^b f(x)f_n(x) dx$ be the n^{th} **Fourier coefficient** of f w.r.t. $\{f_n\}$.

Then $\sum_{n=1}^{\infty} a_n f_n$ is the **Fourier series** of f w.r.t. $\{f_n\}$.

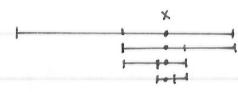
Remarks: (1) $\sum_{n=1}^{\infty} a_n f_n$ does not have to converge!

(2) Even if it converges, it is not always equal to f ! (we cannot assume that this is a good representation for f). odd even

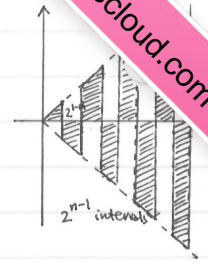
e.g. Take $[-\pi, \pi], \left\{ \frac{1}{\sqrt{\pi}} \cos(nx) \right\}_{n \in \mathbb{N}}$ be the orthonormal system. Take $f(x) = x, a_n = \langle f, f_n \rangle = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x \cos(nx) dx = 0$.

Fourier series is $\sum_{n=1}^{\infty} 0 \cdot f_n = 0 \neq x$.

e.g. Let $\{\varphi_n\}$ be as above. $f(x) = x$ on $[0,1]$. Find its Fourier series: $a_n = \int_0^1 x \varphi_n(x) dx$ are coefficients.
 $a_n = \int_0^1 x \varphi_n(x) dx = -(2^{-n})^2 \cdot 2^{-n-2} = -2^{-2n-2} = -2^{-n}$. $f_S := \sum 2^{-n} \varphi_n(x)$ converges. But then,
 does the Fourier series converge to x ? think of the binary representation of x :



the binary representation of $x = 0.\varphi_1 \dots$
 if $0, \varphi_2 = 1$ if $0, \varphi_2 = 1$
 if $1, \varphi_2 = -1$ if $1, \varphi_2 = -1 \dots$ etc.



Theorem 2.1 (Least squares approximation).

Let $f \in R[a,b]$ and let $\{\varphi_n\}$ be an orthonormal system on $[a,b]$. Denote by a_n the n^{th} Fourier coefficient of f w.r.t. $\{\varphi_n\}$.

then $\|f - \sum_{i=1}^n a_i \varphi_i\|_2 \leq \|f - \sum_{i=1}^n c_i \varphi_i\|_2$ for any n and any sequence $\{c_i\}_{i=1}^n$. The equality holds $\Leftrightarrow a_i = c_i \forall i=1,2,\dots,n$.

Proof - Find $\|f - \sum_{i=1}^n a_i \varphi_i\|_2^2 = \langle f - \sum_{i=1}^n a_i \varphi_i, f - \sum_{j=1}^n a_j \varphi_j \rangle = \langle f, f \rangle - \sum_{j=1}^n a_j \langle f, \varphi_j \rangle - \sum_{i=1}^n a_i \langle f, \varphi_i \rangle + \sum_{i=1}^n \sum_{j=1}^n a_i a_j \langle \varphi_i, \varphi_j \rangle$
 $\therefore \|f - \sum_{i=1}^n a_i \varphi_i\|_2^2 = \langle f, f \rangle - \sum_{j=1}^n a_j^2 - \sum_{i=1}^n a_i^2 + \sum_{k=1}^n a_k^2 = \langle f, f \rangle - \sum_{i=1}^n a_i^2 = \|f\|_2^2 - \sum_{i=1}^n a_i^2$
 $\|f - \sum_{i=1}^n c_i \varphi_i\|_2^2 = \langle f - \sum_{i=1}^n c_i \varphi_i, f - \sum_{j=1}^n c_j \varphi_j \rangle = \langle f, f \rangle - \sum_{j=1}^n c_j \langle f, \varphi_j \rangle - \sum_{i=1}^n c_i \langle f, \varphi_i \rangle + \sum_{i=1}^n \sum_{j=1}^n c_i c_j \langle \varphi_i, \varphi_j \rangle$
 $\therefore \|f - \sum_{i=1}^n c_i \varphi_i\|_2^2 = \langle f, f \rangle - \sum_{j=1}^n c_j a_j - \sum_{i=1}^n a_i c_i + \sum_{i=1}^n c_i^2 = \langle f, f \rangle - \sum_{i=1}^n a_i^2 + \sum_{i=1}^n a_i^2 - 2 \sum_{i=1}^n a_i c_i + \sum_{i=1}^n c_i^2$
 $= \langle f, f \rangle - \sum_{i=1}^n a_i^2 + \sum_{i=1}^n [a_i^2 - 2a_i c_i + c_i^2] = \|f - \sum_{i=1}^n a_i \varphi_i\|_2^2 + \sum_{i=1}^n (a_i - c_i)^2 \geq \|f - \sum_{i=1}^n a_i \varphi_i\|_2^2$

with equality $\Leftrightarrow a_i = c_i \forall i$. Take square roots to get $\|f - \sum_{i=1}^n a_i \varphi_i\|_2 \leq \|f - \sum_{i=1}^n c_i \varphi_i\|_2$.

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Theorem 2.2 (Bessel's inequality)

$\sum_{n=1}^{\infty} a_n^2 \leq \|f\|_2^2$. In particular, $a_n \rightarrow 0$.

Proof - $\|f - \sum_{i=1}^n a_i \varphi_i\|_2^2 = \|f\|_2^2 - \sum_{i=1}^n a_i^2$ (see the proof of theorem 2.1). Thus $\sum_{i=1}^n a_i^2 \leq \|f\|_2^2 \Rightarrow \sum_{i=1}^{\infty} a_i^2$ converges and $\sum_{i=1}^{\infty} a_i^2 \leq \|f\|_2^2$.
 $\sum_{n=1}^{\infty} a_n^2 < \infty \Rightarrow a_n^2 \rightarrow 0 \Leftrightarrow a_n \rightarrow 0$.

Trigonometric Fourier series.

$[-\pi, \pi]$, $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx)\}_{n \in \mathbb{N}}$. $a_n = \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{\pi}} \cos(nx) dx$, $b_n = \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{\pi}} \sin(nx) dx$, $a_0 = \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{2\pi}} dx$.

Fourier series is $\frac{a_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\pi}} \cos(nx) + \sum_{n=1}^{\infty} \frac{b_n}{\sqrt{\pi}} \sin(nx) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{n=1}^{\infty} \frac{1}{\pi} (\int_{-\pi}^{\pi} f(t) \cos(nt) dt) \cos(nx) + \sum_{n=1}^{\infty} \frac{1}{\pi} (\int_{-\pi}^{\pi} f(t) \sin(nt) dt) \sin(nx)$.

Orthogonal system: $\{1, \cos(nx), \sin(nx)\}$. take $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$. then $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$

is the trigonometric Fourier series of f .

Ex Find the trigonometric Fourier series of $f(x) = x$ on $[-\pi, \pi]$.

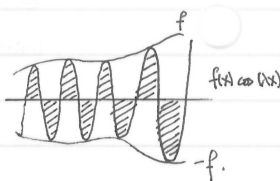
Sol. $a_n = 0$, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = \dots = \frac{2(-1)^{n+1}}{n}$. Then Fourier series is $f(x) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin(nx)$.

Theorem 2.3 (Riemann's Lemma)

Let $f \in R[a,b]$. $\int_a^b f(x) \cos(\lambda x) dx \rightarrow 0$ as $\lambda \rightarrow \infty$. Likewise, $\int_a^b f(x) \sin(\lambda x) dx \rightarrow 0$.

Proof - (i) Let f be a simple step function, that is, \exists partition $P = \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$ s.t. $f(x) = c_i$ if $x \in (t_{i-1}, t_i]$.

Then $|\int_a^b f(x) \cos(\lambda x) dx| \leq \sum_{i=1}^n |c_i| \int_{t_{i-1}}^{t_i} |\cos(\lambda x)| dx = \sum_{i=1}^n |c_i| \frac{|\sin(\lambda t_i) - \sin(\lambda t_{i-1})|}{\lambda} = \sum_{i=1}^n |c_i| \frac{|\sin(\lambda t_i) - \sin(\lambda t_{i-1})|}{\lambda}$
 $\leq \frac{1}{\lambda} \sum_{i=1}^n |c_i| \cdot 2 \rightarrow 0$ as $\lambda \rightarrow \infty$. q.e.d.

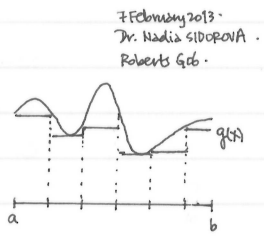


(ii) then we generalize, let $f \in R[a,b] \Rightarrow \forall \epsilon > 0, \exists P \in \mathcal{P}[a,b]$ s.t. $U(f,P) - L(f,P) < \frac{\epsilon}{2}$.

Choose $g(x) = \inf_{t \in [t_{i-1}, t_i]} f(t)$ $x \in (t_{i-1}, t_i]$. then by (i), $\int_a^b g(x) \cos(\lambda x) dx \rightarrow 0$. $\Rightarrow \exists \lambda_0, \forall \lambda \geq \lambda_0$,

$|\int_a^b g(x) \cos(\lambda x) dx| < \frac{\epsilon}{2}$. Then $|\int_a^b f(x) \cos(\lambda x) dx| \leq |\int_a^b g(x) \cos(\lambda x) dx| + |\int_a^b (f(x) - g(x)) \cos(\lambda x) dx|$
 $\Rightarrow \leq \int_a^b |f(x) - g(x)| dx + \frac{\epsilon}{2} = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (f(x) - g(x)) dx + \frac{\epsilon}{2} = \sum_{i=1}^n (f(t_i) - \inf_{t \in [t_{i-1}, t_i]} f(t)) (t_i - t_{i-1}) + \frac{\epsilon}{2}$.

then on $(t_i, t_{i+1}]$, $f(x) \leq \sup_{t \in [t_i, t_{i+1}]} f(t)$, so $|\int_a^b f(x) \cos(\lambda x) dx| \leq \frac{\epsilon}{2} + \sum_{i=1}^n (\sup_{t \in [t_i, t_{i+1}]} f(t) - \inf_{t \in [t_{i-1}, t_i]} f(t)) (t_i - t_{i-1})$
 $= U(f,P) - L(f,P) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. then $\int_a^b f(x) \cos(\lambda x) dx \rightarrow 0$ q.e.d.



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statement about $\sin(\lambda x)$ follows in an analogous approach, q.e.d.

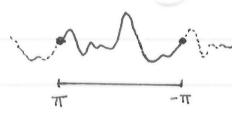
Definition We denote $S_n^f(x)$ by $S_n^f(x) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos(kx) + b_k \sin(kx))$. then $S_n^f(x)$ is the n^{th} partial sum of the trigonometric Fourier series of f .

Theorem 2.4 (Dirichlet's Theorem).

Let $f \in R[-\pi, \pi]$, assume $f(-\pi) = f(\pi)$. Extend f to \mathbb{R} 2π -periodically and denote the extended function by same letter.

then $S_n^f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t) dt \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t) dt$; where $D_n(t) = \begin{cases} \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{t}{2})} & t \neq 2\pi m \\ 1 + 2n & t = 2\pi m \end{cases}$.

and is known as the Dirichlet kernel.



Proof - $S_n^f(x) = \frac{a_0}{2} + \sum_{k=1}^n a_k \cos(kx) + b_k \sin(kx) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{k=1}^n \left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt \right] \cos(kx) + \left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt \right] \sin(kx)$
 $= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[1 + 2 \sum_{k=1}^n (\cos(kx) \cos(kt) + \sin(kx) \sin(kt)) \right] dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[1 + 2 \sum_{k=1}^n \cos(k(x-t)) \right] dt.$

We look specifically at $(1 + 2 \sum_{k=1}^n \cos(k\theta)) \sin \frac{\theta}{2} = \sin \frac{\theta}{2} + \sum_{k=1}^n (\sin[k\theta + \frac{\theta}{2}] - \sin[k\theta - \frac{\theta}{2}]) = \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} - \sin \frac{\theta}{2} + \sin \frac{5\theta}{2} - \sin \frac{3\theta}{2} + \dots$

All terms cancel except $\sin(n\theta + \frac{\theta}{2}) \Rightarrow 1 + 2 \sum_{k=1}^n \cos(k\theta) = \frac{\sin((n+\frac{1}{2})\theta)}{\sin(\theta/2)}$ for $\theta \neq 2\pi m$; $\theta = 2\pi m$ otherwise.

Then $1 + 2 \sum_{k=1}^n \cos(k\theta) = D_n(\theta)$, so $\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[1 + 2 \sum_{k=1}^n \cos(k(x-t)) \right] dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t) dt$ q.e.d.

To get second formula, perform change of variables: $s = x-t$, then $S_n^f(x) = \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x-s) D_n(s) ds$ (-) = $\frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x-s) D_n(s) ds.$

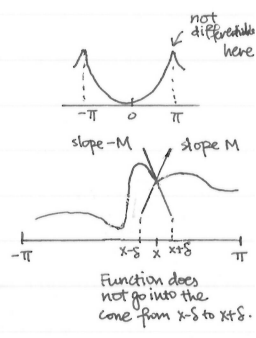
Since function f is 2π -periodic, $S_n^f(x) = \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x-s) D_n(s) ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t) dt$ q.e.d.

Theorem 2.5 (Convergence of Trigonometric Fourier series: I)

Let $f \in R[-\pi, \pi]$ s.t. $f(-\pi) = f(\pi)$, and extend it 2π -periodically to \mathbb{R} . Consider $x \in [-\pi, \pi]$. If f is differentiable at x , then $S_n^f(x) \rightarrow f(x)$ as $n \rightarrow \infty$.

Examples: ① $f(x) = x^2$. We don't know if Fourier series converges at $-\pi, \pi$.

② $f(x) = |x|$. We do not know if Fourier series converges at 0.



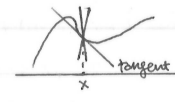
Theorem 2.6 (Convergence of Trigonometric Fourier series: II)

Let $f \in R[-\pi, \pi]$ s.t. $f(-\pi) = f(\pi)$, and extend it 2π -periodically to \mathbb{R} . Let $x \in [-\pi, \pi]$.

Suppose $\exists M > 0$ and $\delta > 0$ s.t. $|f(x+t) - f(x)| \leq M|t| \forall t \in (-\delta, \delta)$. Then $S_n^f(x) \rightarrow f(x)$ as $n \rightarrow \infty$.

Example: If f is differentiable at x , then we note that $S_n^f(x) \rightarrow f(x)$ because it is just

a restricted case of theorem 2.6.



Proof - Note that $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 + 2 \sum_{k=1}^n \cos(kt) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dt = \frac{1}{2\pi} (2\pi) = 1$. Then we have:

$|S_n^f(x) - f(x)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t) dt - f(x) \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t) dt \right| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x-t) - f(x)) D_n(t) dt \right| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} \frac{f(x-t) - f(x)}{\sin(t/2)} \sin((n+\frac{1}{2})t) dt \right|$

We can ignore the finitely many points where $t = 2\pi m$. We denote $g(t) = \frac{f(x-t) - f(x)}{\sin(t/2)}$ for fixed x . We cannot use Riemann's lemma since g may be non-integrable around 0. We examine $g(t)$: $|f(x-t) - f(x)| \leq M|t|$ if $t \in (-\delta, \delta)$. $|\sin \frac{t}{2}| \geq \frac{|t|}{4}$ from graph.

then $|g(t)| = \frac{|f(x-t) - f(x)|}{|\sin(t/2)|} \leq \frac{M|t|}{|t|/4} = 4M$ if $|t| < \min\{\delta, \pi\}$.

Let $\epsilon > 0$. Choose $g_\epsilon(t) = \begin{cases} g(t) & \text{if } |t| > \frac{\epsilon}{2M} \\ 0 & \text{otherwise.} \end{cases}$ then we have, from earlier,

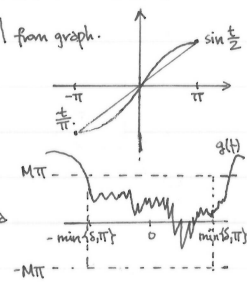
$|S_n^f(x) - f(x)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} g(t) \sin((n+\frac{1}{2})t) dt \right| \leq \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (g(t) - g_\epsilon(t)) \sin((n+\frac{1}{2})t) dt \right| + \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} g_\epsilon(t) \sin((n+\frac{1}{2})t) dt \right|$

By Riemann's lemma, we have that: $\int_{-\pi}^{\pi} (g(t) - g_\epsilon(t)) \sin((n+\frac{1}{2})t) dt \rightarrow 0$ in fact never, except on $[\frac{\epsilon}{2M}, \frac{\epsilon}{2M}]$. Riemann integrable, can evaluate by Riemann's lemma

Since g_ϵ is Riemann integrable, $\left| \int_{-\pi}^{\pi} g_\epsilon(t) \sin((n+\frac{1}{2})t) dt \right| \rightarrow 0$ i.e. $\exists N$ s.t. $\forall n > N$, $\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g_\epsilon(t) \sin((n+\frac{1}{2})t) dt \right| < \frac{\epsilon}{2}$.

Consider $\frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (g(t) - g_\epsilon(t)) \sin((n+\frac{1}{2})t) dt \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t) - g_\epsilon(t)| dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t)| dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} M dt = \frac{M}{2} \cdot \frac{\epsilon}{M} = \frac{\epsilon}{2}$.

Hence overall, $|S_n^f(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \forall n > N \Rightarrow S_n^f(x) \rightarrow f(x)$ as $n \rightarrow \infty$ q.e.d.



Remark: this allows us to prove theorem 2.5 based on our groundwork done.

(Theorem 2.5) Proof - We want to show that f is differentiable at $x \Rightarrow \exists M, \delta$ s.t. $|f(x+t) - f(x)| \leq M|t| \forall t \in (-\delta, \delta)$.

f differentiable at $x \Rightarrow \lim_{t \rightarrow 0} \frac{f(x+t) - f(x)}{t}$ exists. Take $\epsilon = 1 \Rightarrow \exists \delta > 0$ s.t. $\left| \frac{f(x+t) - f(x)}{t} - f'(x) \right| < 1$ whenever $|t| < \delta$.

$\Rightarrow f'(x) - 1 \leq \frac{f(x+t) - f(x)}{t} \leq f'(x) + 1$ for $|t| < \delta$. $f'(x) + 1$ is a number, so $\exists M$ s.t. $\left| \frac{f(x+t) - f(x)}{t} \right| \leq M$ for $|t| < \delta$.

By theorem 2.6, we get $S_n^f(x) \rightarrow f(x)$ for $n \rightarrow \infty$ q.e.d.

However, these theorems don't apply to all functions - there are examples, such as $f(x) = \sqrt{|x|}$. We cannot apply theorems 2.5 and 2.6 at $x=0$.

Take note that this does not necessarily imply that Fourier series at $x=0$ does not converge!

Theorem 2.7 Let $f \in R[-\pi, \pi]$ and $f(-\pi) = f(\pi)$. Suppose f'' exists and $f'' \in R[-\pi, \pi]$. Then $S_n^f \rightarrow f$ uniformly on $[-\pi, \pi]$.

Proof - $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \frac{\sin(nx)}{n} dx \right]_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin(nx) dx = 0 - \frac{1}{n\pi} \left[f'(x) (-\cos nx) \right]_{-\pi}^{\pi} + \frac{1}{n\pi} \int_{-\pi}^{\pi} f''(x) (-\cos nx) dx = \frac{1}{n^2\pi} \int_{-\pi}^{\pi} f''(x) \cos nx dx$

Then $|a_n| \leq \frac{1}{n^2\pi} \int_{-\pi}^{\pi} |f''(x) \cos nx| dx \leq \frac{1}{n^2\pi} \int_{-\pi}^{\pi} |f''(x)| dx = \frac{C}{n^2}$, setting $C = \frac{1}{\pi} \int_{-\pi}^{\pi} |f''(x)| dx$. Likewise, we have that $|b_n| \leq \frac{C}{n^2}$.

As such, we manipulate to get $|S_n^f(x) - f(x)| = \left| \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \right| \leq \sum_{k=1}^{\infty} (|a_k| |\cos(kx)| + |b_k| |\sin(kx)|) \leq \sum_{k=1}^{\infty} (|a_k| + |b_k|)$

thus, $|S_n^f(x) - f(x)| \leq \sum_{k=1}^{\infty} (|a_k| + |b_k|) \leq \sum_{k=1}^{\infty} \frac{C}{k^2} + \frac{C}{k^2} = 2C \sum_{k=1}^{\infty} \frac{1}{k^2}$. Then $\|S_n^f(x) - f(x)\|_{\sup} \leq 2C \sum_{k=1}^{\infty} \frac{1}{k^2} \rightarrow 0$, since $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$.

$\therefore S_n^f \rightarrow f$ uniformly q.e.d.

This marks the end of this chapter.

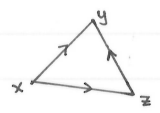
18 February 2013
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Chapter 3:
METRIC SPACES.

When we say $x_n \rightarrow x$, we mean that as the distance between x_n and x tends to 0, $|x_n - x| \rightarrow 0$. Also, $f_n \rightarrow f$ uniformly means as distance between f_n and f tends to 0, $\|f_n - f\|_{\infty} \rightarrow 0$.
Likewise, we can extend this concept further.

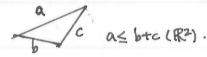
Definition A pair (X, d) of a set X and a function $d: X \times X \rightarrow \mathbb{R}$ is called a metric space if:

- (i) $d(x, y) \geq 0 \quad \forall x, y \in X$ and $d(x, y) = 0 \Rightarrow x = y$.
- (ii) $d(x, y) = d(y, x) \quad \forall x, y \in X$.
- (iii) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$ (triangle inequality).



d is called a metric function (or distance function).

e.g. ① \mathbb{R} , $d(x, y) = |x - y|$. ② \mathbb{R}^n , $d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ (Euclidean distance).



③ $C[a, b]$ i.e. continuous functions on $[a, b]$, $d(f, g) = \|f - g\|_{\infty}$.

④ Discrete space: any set X , $d(x, y) = \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases}$. We check conditions for metric space. (iii): let $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$
 case 1: $x=y$, then $d(x, y) = 0 \leq d(x, z) + d(z, y)$. case 2: $x \neq y$, then $d(x, y) = 1 \leq 1 + 0 = 1$ or $1 + 1 = 2$.

⑤ "British railway metric": let X be a set, specify $L \in X$. let $f(x)$ be the distance from x to L .
 then $f(x) \geq 0, f(x) = 0 \Leftrightarrow x = L$. We define $d(x, y) = \begin{cases} f(x) + f(y) & x \neq y \\ 0 & x = y \end{cases}$.



Definition A pair of a vector space V and a norm function $\|\cdot\|: V \rightarrow \mathbb{R}$ is called a normed space if:

- (i) $\|x\| \geq 0 \quad \forall x \in V$, $\|x\| = 0 \Leftrightarrow x = 0$.
- (ii) $\|\lambda x\| = |\lambda| \|x\| \quad \forall x \in V, \lambda \in \mathbb{R}$.
- (iii) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$.

Theorem 3.1 Every normed space $(V, \|\cdot\|)$ can be made into a metric space (V, d) with $d(x, y) = \|x - y\|$.

Proof - (i) $d(x, y) = \|x - y\| \geq 0$, $d(x, y) = 0 \Rightarrow \|x - y\| = 0 \Rightarrow x = y$. (ii) $d(x, y) = \|x - y\| = \|(y - x)\| = |(-1)| \|y - x\| = \|y - x\| = d(y, x)$.
 (iii) $d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y)$ q.e.d.

Remark: Not every metric space is a normed space: for instance, (X, d) discrete metric space: $d(x, y) = \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases}$.

X is not necessarily a vector space, and even if it did, it would not apply.

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e.g. ① \mathbb{R} , $\|x\| = |x|$ is a normed space with metric $d(x, y) = \|x - y\| = |x - y|$ corresponding to it.

② \mathbb{R}^n , with Euclidean norm $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ is a normed space. Likewise, we have 1-norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$.

of course, we can generalise this to the q -norm $\|x\|_q = \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}}$. Proving this is a norm is difficult, particularly the triangle inequality.

$\|x\|_{\infty} = \max_{1 \leq i \leq n} |x_i|$ is the infinity-norm.

The corresponding metrics are $d_2(x, y) = \|x - y\|_2 = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$, $d_1(x, y) = \|x - y\|_1 = \sum_{i=1}^n |x_i - y_i|$, $d_q(x, y) = \|x - y\|_q = \left(\sum_{i=1}^n |x_i - y_i|^q\right)^{\frac{1}{q}}$.

Also, $d_{\infty}(x, y) = \|x - y\|_{\infty} = \max_{1 \leq i \leq n} |x_i - y_i|$.

③ $C[a, b]$. let $\|f\|_{\infty} = \sup_{x \in [a, b]} |f(x)|$ be the supremum norm. (corresponds to infinity norm in ②). 2-norm is $\|f\|_2 = \sqrt{\int_a^b f(x)^2 dx}$.

Also, we have q -norm $\|f\|_q = \left(\int_a^b |f(x)|^q dx\right)^{\frac{1}{q}}$, 1-norm $\|f\|_1 = \int_a^b |f(x)| dx$. Metrics are found analogously.

Theorem 3.2 $\|\cdot\|_{\infty}$ and $\|\cdot\|_2$ are norms on $C[a, b]$.

Proof - (a) For $\|\cdot\|_{\infty}$: (i) $\|f\|_{\infty} \geq 0$ obvious. $\sup_{[a, b]} |f(x)| = 0 \Rightarrow f(x) = 0 \quad \forall x \Leftrightarrow f = 0$. (ii) $\|\lambda f\|_{\infty} = \sup_{[a, b]} |\lambda f(x)| = |\lambda| \sup_{[a, b]} |f(x)| = |\lambda| \|f\|_{\infty}$.
 (iii) $\|f + g\|_{\infty} = \sup_{[a, b]} |f(x) + g(x)| \leq \sup_{[a, b]} (|f(x)| + |g(x)|) \leq \sup_{[a, b]} |f(x)| + \sup_{[a, b]} |g(x)| = \|f\|_{\infty} + \|g\|_{\infty}$ q.e.d.

(b) For $\|\cdot\|_2$: (i) $\|f\|_2 \geq 0$ \because it is a square root. $\|f\|_2 = 0 \Leftrightarrow \int_a^b f(x)^2 dx = 0 \Leftrightarrow f = 0$ (since f is continuous).

(ii) $\|\lambda f\|_2 = \sqrt{\int_a^b (\lambda f(x))^2 dx} = |\lambda| \sqrt{\int_a^b f(x)^2 dx} = |\lambda| \|f\|_2$.

(iii) $\|f + g\|_2^2 = \langle f + g, f + g \rangle = \|f\|_2^2 + 2\langle f, g \rangle + \|g\|_2^2$. By Cauchy-Schwarz inequality, $|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2$.

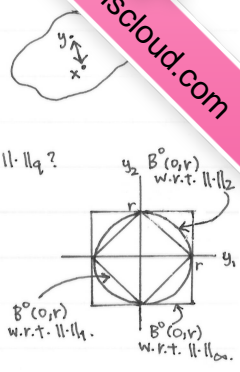
then $\|f + g\|_2^2 \leq \|f\|_2^2 + 2\|f\|_2 \|g\|_2 + \|g\|_2^2 = (\|f\|_2 + \|g\|_2)^2$.

Hence, we see that $\|f + g\|_2 \leq (\|f\|_2 + \|g\|_2)^2$. since both must have the square roots,

$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$ q.e.d.

Definition let (X, d) be a metric space. let $x \in X$ and $r > 0$. then $B^o(x, r) = \{y \in X : d(y, x) < r\}$ is called the open ball of radius r with centre at x . Similarly, $B(x, r) = \{y \in X : d(y, x) \leq r\}$ is the closed ball of radius r with centre at x .

e.g. ① $(\mathbb{R}, |\cdot|)$: $B^o(x, r) = \{y \in \mathbb{R} : |y-x| < r\} = (x-r, x+r)$. $B(x, r) = [x-r, x+r]$.

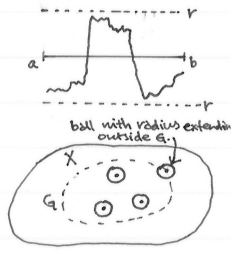


- ② \mathbb{R}^2 . What are the open balls and closed balls $B^o(0, r)$, $B(0, r)$ with respect to the metrics $\|\cdot\|_2$, $\|\cdot\|_1$, $\|\cdot\|_\infty$, $\|\cdot\|_q$?
- For $\|\cdot\|_2$, $B^o(0, r) = \{y \in \mathbb{R}^2 : \|y\|_2 < r\}$ i.e. $y_1^2 + y_2^2 < r^2 \Rightarrow$ circle of radius r .
- For $\|\cdot\|_\infty$, $B^o(0, r) = \{y \in \mathbb{R}^2 : \|y\|_\infty < r\}$ i.e. $\max\{|y_1|, |y_2|\} < r$ i.e. $|y_1|, |y_2| < r$.
- For $\|\cdot\|_1$, $B^o(0, r) = \{y \in \mathbb{R}^2 : \|y\|_1 < r\}$ i.e. $|y_1| + |y_2| < r \Rightarrow |y_2| < r - |y_1| \Rightarrow$ diamond (as shown)

Heuristically, we think about $\|\cdot\|_q$, with $q=3, 4, 5, \dots$ we will get something in between the shape for $\|\cdot\|_2$ and $\|\cdot\|_\infty$. To get closed balls $B(0, r)$, we take $B^o(0, r) + \partial B^o(0, r)$, its boundary.

- ③ Discrete space: set X with $d(x, y) = \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases}$. $B^o(x, r) = \{y \in X : d(y, x) < r\} = \begin{cases} X, & r > 1 \\ \{x\}, & r \leq 1 \end{cases}$.
- $B(x, r) = \{y \in X : d(y, x) \leq r\} = \begin{cases} X, & r \geq 1 \\ \{x\}, & r < 1 \end{cases}$ (note the difference in inequality signs).

- ④ $C[a, b]$ with $\|\cdot\|_{\text{sup}}$. $B^o(\text{zero function}, r) = \{f : \|f\|_{\text{sup}} < r\} = \{\text{all functions remaining completely and strictly inside the } r\text{-tube around the zero function}\}$.



Definition let (X, d) be a metric space. A set $G \subset X$ is called an open set if $\forall x \in G, \exists r > 0$ s.t. $B^o(x, r) \subset G$.

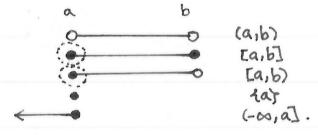
A set $F \subset X$ is called a closed set if its complement $X \setminus F$ is open.

Remarks: there are sets which are neither open nor closed. More interestingly, there are sets which are open AND closed at the same time.

Hence, we cannot prove that a set is open by showing that it is not closed (and vice versa).

e.g. ① $(\mathbb{R}, |\cdot|)$: then we examine the following sets:

	(a, b)	$[a, b]$	$[a, b)$	$(a, b]$	$(-\infty, a]$
set is open?	✓	X	X	X	X
set is closed?	X	✓	X	✓	✓



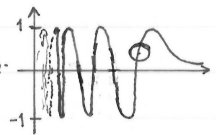
then consider $\{\frac{1}{n} : n \in \mathbb{N}\}$. the set is made of discrete points, so it is not open. It is not closed either, as 0 in complement is a "bad" point since r is a fixed value.

However, while $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is also not open, it is closed.

- ② Take \mathbb{R}^2 with $\|\cdot\|_2$. Consider $B^o(0, r)$. it is clearly an open set, but not a closed set. On the other hand, $B(0, r)$ is not an open set, but is a closed set.
- $A = \{(x, \sin \frac{1}{x}) : x > 0\} \subset \mathbb{R}^2$.

the set is obviously not open. Take any point lying on the graph itself. a ball of any radius > 0 contains points not within G .

consider $\mathbb{R}^2 \setminus A$. Then at $x=0$, the set is not open \Rightarrow set A is not closed.



Note then that the set $F = \{(x, \sin \frac{1}{x}), x > 0\} \cup \{(0, y) : -1 \leq y \leq 1\}$ is still not open, but is closed.

- ② Let (X, d) be a discrete space, $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$. $B^o(x, r) = \begin{cases} \{x\}, & r \leq 1 \\ X, & r > 1 \end{cases}$. Let $S \subset X$.

If S is open, $\forall x \in S$, take $r \leq 1$. then $B^o(x, r) = \{x\} \subset S \Rightarrow$ any set in a discrete metric space is open!

likewise, any set in a discrete metric space is closed.



Theorem 3.3

(a) Every open ball $B^o(x, r)$ is an open set.

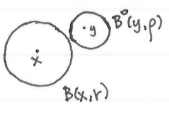
(b) Every closed ball $B(x, r)$ is a closed set.

Proof- (a). let $y \in B^o(x, r) \Rightarrow d(x, y) < r$. Take $p = r - d(x, y) > 0$. Consider $B^o(y, p)$. We need to prove that $B^o(y, p) \subset B^o(x, r)$.

Pick $z \in B^o(y, p)$. then $|z-y| < p$. Check $d(x, z) = |x-z| < |x-y| + |y-z| = d(x, y) + d(y, z) < d(x, y) + p = r \Rightarrow z \in B^o(x, r)$ q.e.d.

(b). Consider $B(x, r)$. let $y \in X \setminus B(x, r) \Rightarrow d(x, y) > r$. Take $p = d(x, y) - r$, and consider $B^o(y, p)$. We want to prove that:

$B^o(y, p) \subset X \setminus B(x, r)$. Pick $z \in B^o(y, p)$, then $d(y, z) < p$. $d(x, z) \geq d(x, y) - d(y, z) > d(x, y) - p = r \Rightarrow z \in X \setminus B(x, r)$ q.e.d.



Theorem 3.4 let (X, d) be a metric space. Then

- (a) \emptyset and X are both open and closed.
- (b) let $\{G_\alpha\}_{\alpha \in A}$ be a collection of open sets. then $\bigcup_{\alpha \in A} G_\alpha$ is an open set.
- (c) let $\{G_i\}_{i=1}^n$ (finite n) be a collection of open sets. then $\bigcap_{i=1}^n G_i$ is an open set.
- (d) let $\{F_\alpha\}_{\alpha \in A}$ be a collection of closed sets. then $\bigcap_{\alpha \in A} F_\alpha$ is a closed set.
- (e) let $\{F_i\}_{i=1}^n$ be a finite collection of closed sets. then $\bigcup_{i=1}^n F_i$ is a closed set.

Remark: what is wrong with (c) and (e) for infinite collections? We exhibit this through some examples.

e.g. For (c), consider $G_n = (-\frac{1}{n}, \frac{1}{n})$; which are open sets. then $\bigcap_{n=1}^{\infty} G_n = \{0\}$, which is not open. Also, for (e), consider closed sets $F_n = [-\frac{1}{n}, \frac{1}{n}]$, $n \geq 2$. then $\bigcup_{n=2}^{\infty} F_n = (-1, 1)$, which is not closed.

Proof - (a). X is open, since $\forall x \in X, B^o(x, r) \subset X$ for any r . ϕ is open, since there are no points for which we should check anything. Or, try the following: G is open means $\forall x \in G, \exists r B^o(x, r) \subset G$. G is not open means $\exists x \in G, \forall r > 0 B^o(x, r) \not\subset G$. But $\exists x \in \phi$ at all, so ϕ is not not open $\Rightarrow \phi$ is open. q.e.d. Then for closure, we note the trivial observation that X, ϕ are merely complements of each other. then X is closed $\Leftrightarrow \phi$ is open, ϕ is closed $\Leftrightarrow X$ is open, q.e.d.

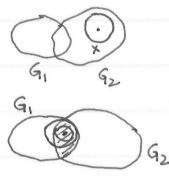
(b) let $x \in \bigcup_{\alpha \in A} G_\alpha$. then $x \in G_{\alpha_0}$ for some $\alpha_0 \in A$. since G_{α_0} is open, $\exists r > 0$ s.t. $B^o(x, r) \subset G_{\alpha_0}$. However, since $\alpha_0 \in A, B^o(x, r) \subset G_{\alpha_0} \subset \bigcup_{\alpha \in A} G_\alpha \Rightarrow \bigcup_{\alpha \in A} G_\alpha$ is open, q.e.d.

(c) let $x \in \bigcap_{i=1}^n G_i$. then $x \in G_1, x \in G_2, \dots, x \in G_n$. since all $\{G_i\}_{i=1}^n$ are open, for each $1 \leq i \leq n, \exists r_i > 0$ s.t. $B^o(x, r_i) \subset G_i$. then let $r = \min_{1 \leq i \leq n} r_i$. then $B^o(x, r) \subset B^o(x, r_i) \subset G_i \forall i$. since $B^o(x, r) \subset G_i$ for all $G_i, B^o(x, r) \subset \bigcap_{i=1}^n G_i \Rightarrow \bigcap_{i=1}^n G_i$ is open, q.e.d.

(d) All F_α are closed $\Rightarrow X \setminus F_\alpha$ are open \Rightarrow by part (b), $\bigcup_{\alpha \in A} (X \setminus F_\alpha)$ is open $\Rightarrow X \setminus (\bigcap_{\alpha \in A} F_\alpha)$ is open $\Rightarrow \bigcap_{\alpha \in A} F_\alpha$ is closed, q.e.d.

(e) All $\{F_i\}_{i=1}^n$ are closed \Rightarrow all $X \setminus F_i$ are open \Rightarrow by part (c), $\bigcap_{i=1}^n (X \setminus F_i)$ is open $\Rightarrow X \setminus (\bigcup_{i=1}^n F_i)$ is open $\Rightarrow \bigcup_{i=1}^n F_i$ is closed, q.e.d.

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Definition let (X, d) be a metric space. We say that a sequence of points in X converges to $x \in X$ (we write $x_n \rightarrow x$ as $n \rightarrow \infty$) if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. Equivalently, $x_n \rightarrow x \Leftrightarrow \forall \epsilon > 0, \exists N \forall n \geq N, d(x_n, x) < \epsilon$. (or $x_n \in B^o(x, \epsilon)$.)

e.g. ① $(\mathbb{R}, || \cdot ||)$. Then $x_n \rightarrow x \Leftrightarrow |x_n - x| \rightarrow 0 \Leftrightarrow$ convergent sequences are the usual convergent sequences in \mathbb{R} .

② $(0, 1), d$, where $d(x, y) = |x - y|$. let $x_n = \frac{1}{n}$. then $\frac{1}{n} \rightarrow 0$ number strictly between 0 and 1? No $\Rightarrow \{1/n\}$ does not converge.

However, $y_n = \frac{1}{2} + \frac{1}{n} \rightarrow \frac{1}{2}$ converges in this space.

③ Discrete space, (X, d) , then $x_n \rightarrow x \Leftrightarrow d(x_n, x) \rightarrow 0$. $d(x, x_i): \underbrace{* \quad * \quad * \quad * \quad \dots}_{\text{any values}} \quad \dots \quad 0 \quad 0 \quad 0 \quad 0 \rightarrow 0$.

A sequence $\{x_n\}$ converges if it is eventually constant.

④ $(C[a, b], || \cdot ||_{sup})$. then $f_n \rightarrow f \Leftrightarrow d(f_n, f) \rightarrow 0$ i.e. $||f_n - f||_{sup} \rightarrow 0$. True $\Leftrightarrow f_n \rightarrow f$ uniformly on $[a, b]$.



Theorem 3.5 If $x_n \rightarrow x$ and $x_n \rightarrow y$, then $x = y$.

Proof - Assume $x \neq y$. Then $d(x, y) > 0 < d(x, y) \leq d(x, x_n) + d(x_n, y) \rightarrow 0 + 0 = 0 \Rightarrow 0 < 0$, which is a contradiction.

Hence, assumption is wrong $\Rightarrow x = y$, q.e.d.

Recall some motivating examples: ① $(\mathbb{R}, || \cdot ||)$. $A = \{1/n : n \in \mathbb{N}\}$ is not closed. ② $(\mathbb{R}^2, \text{Euclidean } d)$. $(x, y) = (x, \sin \frac{1}{x})$.

Theorem 3.6 Let (X, d) be a metric space and $A \subset X$. A is closed \Leftrightarrow If sequence $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \in A \forall n$ converges, then it converges to point $x \in A$.

Proof - (\Rightarrow). Suppose A is closed, take $x_n \in A \forall n, x_n \rightarrow$ some x . NTP: $x \in A$.

By contradiction. Suppose $x \notin A, x \in X \setminus A$. A is closed $\Rightarrow X \setminus A$ is open. then $\exists r > 0$ s.t. $B^o(x, r) \subset X \setminus A$

then $x_n \rightarrow x \Rightarrow \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, d(x_n, x) < r \Rightarrow x_n \in B^o(x, r) \subset X \setminus A \Rightarrow x_n \notin A$.

But $x_n \in A$ by definition \Rightarrow contradiction.

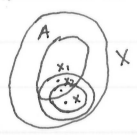
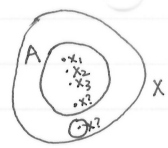
(\Leftarrow). Suppose RHS is true, but assume A is not closed. $\Rightarrow X \setminus A$ is not open. (i.e. not every point can be surrounded by a ball)

\Rightarrow at least one point cannot be surrounded by a ball. $\exists x \in X \setminus A$ s.t. $\forall r > 0, B^o(x, r) \cap A \neq \phi$.

In particular, for $r = \frac{1}{n}, B^o(x, \frac{1}{n}) \cap A \neq \phi \forall n$. i.e. $\exists x_n \in B^o(x, \frac{1}{n}) \cap A$. Then, each $x_n \in A$.

Also, $d(x_n, x) < \frac{1}{n} \rightarrow 0$ i.e. $x_n \rightarrow x$. However, $x \notin A$ so $x \in X \setminus A \Rightarrow$ contradicts RHS. Hence, $X \setminus A$ must be open

$\therefore A$ is closed, q.e.d.



Definition Let (X, d) be a metric space. A sequence $\{x_n\}_{n \in \mathbb{N}}$ of points in X is called a Cauchy sequence if $\forall \epsilon > 0 \exists N \forall n, m \geq N, d(x_n, x_m) < \epsilon$.

e.g. ① $(\mathbb{R}, || \cdot ||)$. Cauchy sequences are the "usual" Cauchy sequences in \mathbb{R} .

② $(0, 1), d$. $d(x, y) = |x - y|$. $x_n = \frac{1}{n}$ is Cauchy: $\forall \epsilon > 0 \exists N \forall n, m \geq N, |\frac{1}{n} - \frac{1}{m}| < \epsilon$.

③ Discrete space, (X, d) . Then x_n is Cauchy $\Leftrightarrow \forall \epsilon > 0 \exists N \forall n, m \geq N, d(x_n, x_m) < \epsilon$. $d(x_n, x_m) = \begin{cases} 1 & x_n \neq x_m \\ 0 & x_n = x_m \end{cases}$

i.e. $\forall n, m \geq N, x_n = x_m$. Hence $\{x_n\}$ is Cauchy $\Leftrightarrow \{x_n\}$ is eventually constant.

④ $(C[a, b], || \cdot ||_{sup})$. $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy $\Leftrightarrow \forall \epsilon > 0 \exists N$ s.t. $\forall n, m \geq N, ||f_n - f_m||_{sup} < \epsilon$. then $\{f_n\}_{n \in \mathbb{N}}$ is a uniform Cauchy sequence.

Remarks: For examples ①, ③, ④, convergence criteria are the same - but they differ for ②. We investigate this further.

Lemma If x_n converges, then x_n is a Cauchy sequence. [Remark: Converse is not true!]

Proof - Suppose $x_n \rightarrow x \Rightarrow \forall \epsilon > 0, \exists N \forall n > N, d(x_n, x) < \frac{\epsilon}{2} \Rightarrow \forall n, m > N, d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ q.e.d.

Definition A metric space is called complete if every Cauchy sequence in that metric space converges.

A complete normed space is called a Banach space.

- e.g. ① $(\mathbb{R}, |\cdot|)$ is a complete space and Banach space.
 ② $(\mathbb{C}, |\cdot|)$ is not complete (and of course not a Banach space).
 ③ the discrete space is complete, but it is not a Banach space because a discrete metric is not a norm.
 ④ $(C[a, b], \|\cdot\|_{\text{sup}})$ is complete by the CUC. since it is a vector space and $\|\cdot\|_{\text{sup}}$ is a norm, it is also a Banach space.
 ⑤ More complete spaces (Banach spaces, in fact): \mathbb{R}^n with $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_{\infty}, \|\cdot\|_p$ for $p \geq 1$.
 • $C[a, b]$ with $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_{\text{sup}}$ or $\|\cdot\|_p, p \geq 1$. (or any irrational p)
 ⑥ the metric space $(\mathbb{Q}, |\cdot|)$ will not be complete. Consider $x_n \in \mathbb{Q}, x_n \rightarrow \sqrt{2}$ as $n \rightarrow \infty$. x_n does not converge in $(\mathbb{Q}, |\cdot|)$.
 However, x_n converges in $\mathbb{R} \Rightarrow x_n$ is Cauchy in $\mathbb{R} \Rightarrow x_n$ is Cauchy in $(\mathbb{Q}, |\cdot|)$. Thus, $(\mathbb{Q}, |\cdot|)$ is not complete.

Lemma Let (X, d) be a complete metric space. then if $Y \subset X, (Y, d)$ is complete $\Leftrightarrow Y$ is closed.

Proof - (\Rightarrow) Suppose (Y, d) is complete. We want: Y is closed (use theorem 3.6). Let $x_n \in Y$, then $x_n \rightarrow x \in X \Rightarrow \{x_n\}$ is Cauchy in (X, d) .

thus, $\{x_n\}$ is Cauchy in (Y, d) . since (Y, d) is complete, $x_n \rightarrow x$ in $Y \Rightarrow x \in Y$ q.e.d.

(\Leftarrow). Suppose Y is closed. We want to show that (Y, d) is complete. Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in (Y, d) .

then $\{x_n\}$ is a Cauchy sequence in $(X, d) \Rightarrow x_n \rightarrow x \in X$ since (X, d) is complete. As Y is closed, $x \in Y \Rightarrow x_n$ converges in (Y, d) q.e.d.

contraction mappings:

Definition Let (X, d) be a metric space. A mapping $T: X \rightarrow X$ is called a contraction mapping if $\exists c \in (0, 1)$ st. $d(Tx, Ty) \leq c \cdot d(x, y) \forall x, y \in X$.

Note: We write Tx for $T(x)$, point of nomenclature.
 is a contraction mapping.

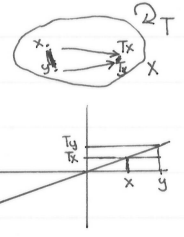
e.g. ① $(\mathbb{R}, |\cdot|), Tx = \frac{1}{2}x$ then $|Tx - Ty| = \frac{1}{2}|x - y| \leq \frac{1}{2}|x - y|, c = \frac{1}{2} \Rightarrow T$ is a contraction with $c = \frac{1}{2}$.
 MVT.

② $(\mathbb{R}, |\cdot|), Tx = \sin(\frac{x}{3})$. then $|Tx - Ty| = |\sin(\frac{x}{3}) - \sin(\frac{y}{3})| = |\frac{1}{3} \cos(\frac{\xi}{3})| |x - y| \leq \frac{1}{3} |x - y|$.

T is a contraction mapping with $c = \frac{1}{3}$.

More generally, for $(\mathbb{R}, |\cdot|), T$ is differentiable $\Rightarrow |Tx - Ty| \stackrel{\text{MVT}}{=} |T'(\xi)| |x - y| \leq \|T'\|_{\text{sup}} |x - y|$. if $\|T'\|_{\text{sup}} < 1$, then T is a contraction mapping.

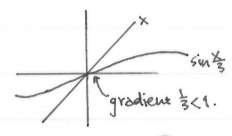
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Definition Let (X, d) be a metric space and $T: X \rightarrow X$. if $x \in X$ has the property $Tx = x$, then x is called a fixed point of T .

e.g. ① $(\mathbb{R}, |\cdot|), Tx = \frac{x}{2}$. $x = 0$ is the unique fixed point.

② $(\mathbb{R}, |\cdot|), Tx = \sin(\frac{x}{3})$. Find $\sin(\frac{x}{3}) = x$. then $x = 0$ is the unique fixed point.



Theorem \Rightarrow T (Contraction Mapping Theorem).

Let (X, d) be a non-empty complete metric space, and let $T: X \rightarrow X$ be a contraction mapping. then T has a unique fixed point.

Proof - Denote the contraction constant of T by c . Since $X \neq \emptyset$, we can pick some $x_0 \in X$. Define $\{x_n\}$ by $x_n = Tx_{n-1}$

$$\forall n \in \mathbb{N}. \text{ then } d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq c d(x_n, x_{n-1}) \leq c^2 d(x_{n-1}, x_{n-2}) \text{ etc... } \leq c^n d(x_1, x_0).$$

$$\text{wlog, assume } m > n. \text{ then } d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{n+1}, x_n) \leq (c^{m-1} + c^{m-2} + \dots + c^{n+1} + c^n) d(x_1, x_0).$$

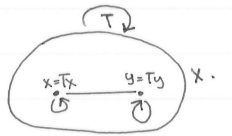
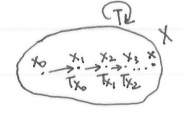
$$\text{i.e. } d(x_m, x_n) \leq \sum_{i=1}^{m-n} c^i d(x_1, x_0) \leq \frac{c^n}{1-c} d(x_1, x_0) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ as } c \in (0, 1).$$

Hence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence \Rightarrow since (X, d) is complete, $\{x_n\}_{n \in \mathbb{N}}$ converges to some $x \in X$.

$x_n \rightarrow x \Rightarrow x_{n+1} \rightarrow x$. then $d(x_{n+1}, Tx) = d(Tx_n, Tx) \leq c d(x_n, x) \rightarrow 0 \Rightarrow x_{n+1} \rightarrow Tx$. Hence, $Tx = x$.

suppose x, y are both fixed points, then $Tx = x, Ty = y$. $|Tx - Ty| = |x - y| = d(x, y) \leq c d(x, y) \Rightarrow c \geq 1$

However, T is a contraction mapping $\Rightarrow c \in (0, 1) \Rightarrow$ contradiction \Rightarrow only one fixed point, q.e.d.

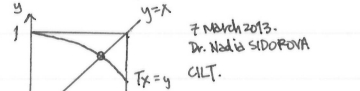


Remark: Why are the assumptions of the CMT important?

① X must be complete: $X = (0, \infty)$ with $d(x, y) = |x - y|$. Take $T(x) = \frac{x}{2}$ is a contraction, but $x = Tx \Leftrightarrow x = \frac{x}{2} \Rightarrow$ no solutions in $(0, \infty)$. T has no fixed point.

② We cannot replace $d(Tx, Ty) \leq c d(x, y)$ by $d(Tx, Ty) < d(x, y)$. The space $X = [1, \infty)$ with $d(x, y) = |x - y|$ is complete (since closed).

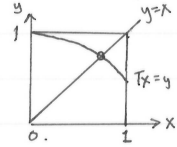
If $Tx = x + \frac{1}{x}$, $|Tx - Ty| = |1 - \frac{1}{\xi}| |x - y| < |x - y|$. $Tx = x \Leftrightarrow x + \frac{1}{x} = x \Rightarrow \frac{1}{x} = 0 \Rightarrow$ no solution $\Rightarrow T$ has no fixed point.



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Applications of the CMT:

① Take $[0, 1], d(x, y) = |x - y|$. $Tx = \cos(x)$. $-T$ is a contraction mapping on $[0, 1]$. We know that $\cos(x)$ is not a contraction mapping on \mathbb{R} .



For $Tx=x$ on $[0,1]$, there is a unique solution: set $x_0=0$, $\cos(x_{n-1})=x_n$. Then by contraction Mapping Theorem, $x_n \rightarrow x$ (fixed point).

This gives us a method to technically compute x s.t. $Tx=x$.

② consider differential equation $y' = f(x,y)$, $y(x_0) = y_0$. For instance, $y' = xy$, $y(0) = 1$. [Clearly: $y(x) = e^{x^2/2}$ in this (easy) case where f is a "nice" function].
In general, we can integrate to get $\int_{x_0}^x y' = \int_{x_0}^x f(t, y(t)) dt \Rightarrow y(x) = y_0 + \int_{x_0}^x f(t, y(t)) dt$. Assume $y(x)$ is continuous [implicit since y differentiable].
Hence, we have an equation $y = Ty$, where the mapping T is s.t. $(T\varphi)(x) = \int_{x_0}^x f(t, \varphi(t)) dt + y_0$. Hence, we are actually looking for a fixed point to mapping T .
one can show that under some reasonable assumptions, T is a contraction mapping (on some suitable space of functions). then solving ② is equivalent to solving ③, which is equivalent to finding a fixed point of T . i.e. Take $\varphi_0(x)$ to be any function, $\varphi_1 = T\varphi_0$, $\varphi_2 = T\varphi_1$, $\varphi_3 = T\varphi_2$, ...

For instance, consider the system $y' = xy$, $y(0) = 1$. Take $\varphi_0(x) = 1$. $\varphi_1(x) = 1 + \int_0^x t \cdot 1 dt = 1 + \frac{1}{2}x^2$. $\varphi_2(x) = 1 + \int_0^x t(\varphi_1(t)) dt = 1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 2}$.
 $\varphi_3(x) = 1 + \int_0^x t \varphi_2(t) dt = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{6 \cdot 8} = 1 + \frac{x^2}{2} + \frac{(\frac{x^2}{2})^2}{2!} + \frac{(\frac{x^2}{2})^3}{3!}$. continuing in this pattern, $\varphi_n(x) = \sum_{i=0}^n \frac{1}{i!} (\frac{x^2}{2})^i$. As $n \rightarrow \infty$, $\varphi_n(x) \rightarrow e^{x^2/2}$ q.e.d.

Theorem 3.8 (Picard theorem).

suppose $f: [a,b] \times [c,d] \rightarrow \mathbb{R}$ is continuous, and $\frac{\partial f}{\partial y}$ exists and is continuous. let $(x_0, y_0) \in (a,b) \times (c,d)$. Then $\exists h_0 > 0$ s.t. ④ $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ has a unique solution on $[x_0-h_0, x_0+h_0]$.

Proof-(existence) WLOG, assume that $a < 0 < b$, $c < 0 < d$ and $x_0 = 0, y_0 = 0$ (by translation of rectangle).

f is continuous on $[a,b] \times [c,d] \Rightarrow f$ is bounded $\Rightarrow |f(x,y)| \leq M$ for some $M, \forall x,y$.

$\frac{\partial f}{\partial y}$ is continuous on $[a,b] \times [c,d] \Rightarrow \frac{\partial f}{\partial y}$ is bounded $\Rightarrow |\frac{\partial f}{\partial y}(x,y)| \leq \tilde{M}$ for some $\tilde{M}, \forall x,y$.

then $\begin{cases} y' = f(x,y) \\ y(0) = 0 \end{cases} \Leftrightarrow \begin{cases} y \text{ is continuous.} \\ \text{choose } h_1, k \text{ s.t. } [-h_1, h_1] \times [-k, k] \subset [a,b] \times [c,d]. \end{cases}$

let $h \leq h_1$. consider $B(h,k) = \{ \varphi: [-h,h] \rightarrow \mathbb{R}, \varphi \text{ is continuous and } \|\varphi\|_{\text{sup}} \leq k \}$.

consider mapping $(T\varphi)(x) = \int_0^x f(t, \varphi(t)) dt$

we will show that $T: B(h,k) \rightarrow B(h,k)$; $\|T\varphi\|_{\text{sup}} = \sup_{x \in [-h,h]} |\int_0^x f(t, \varphi(t)) dt| \leq M \cdot h \leq \frac{k}{2} \leq k$ whenever $h < \frac{k}{2M}$

$B(h,k)$ with $\|\cdot\|_{\text{sup}}$ is complete, $B(h,k) \subset C[-h,h]$ (both with $\|\cdot\|_{\text{sup}}$). $C[-h,h]$ is complete. then $B(h,k)$ is now complete since it is closed: if $\varphi_n \in B(h,k)$ s.t. $\varphi_n \rightarrow \varphi$ uniformly, then $|\varphi(x)| \leq k \forall x \Rightarrow \varphi \in B(h,k)$.

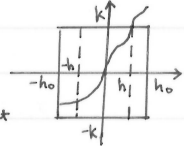
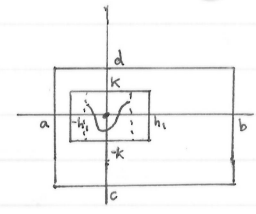
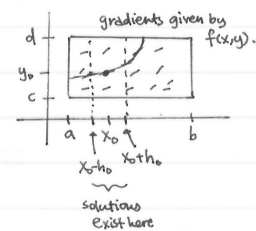
T is a contraction mapping on $B(h,k)$: let $\varphi, \psi \in B(h,k)$. then $\|T\varphi - T\psi\|_{\text{sup}} = \sup_{x \in [-h,h]} |\int_0^x (f(t, \varphi(t)) - f(t, \psi(t))) dt|$
 $\|T\varphi - T\psi\|_{\text{sup}} = \sup_{x \in [-h,h]} |\int_0^x (f(t, \varphi(t)) - f(t, \psi(t))) dt| \stackrel{MVT}{=} \sup_{x \in [-h,h]} |\int_0^x \frac{\partial f}{\partial y}(t, \xi) (\varphi(t) - \psi(t)) dt| \leq h \tilde{M} \|\varphi - \psi\|_{\text{sup}}$
set $h \tilde{M} < 1 \Rightarrow h < \frac{1}{\tilde{M}}$. Choose $h_0 = \min\{h_1, \frac{k}{2M}, \frac{1}{\tilde{M}}\}$. then T is a contraction mapping on complete space $B(h_0, k)$

\therefore By CMT, $T\varphi = \varphi$ has a unique solution \Rightarrow ④ has a unique solution on $[-h_0, h_0]$ among functions which are bounded by k .

It remains to show that there is no solution on $[-h_0, h_0]$ with norm $> k$.

suppose $\exists \varphi^*$ s.t. φ^* solves ④ on $[-h_0, h_0]$ and $\|\varphi^*\|_{\text{sup}} > k$. let $h = \inf\{|t| : |\varphi^*(t)| > k\}$.

Denote $\bar{\varphi}^*$ as the restriction of φ^* to $[-h, h]$. $\bar{\varphi}^*$ is a solution of ④ on $[-h, h] \Rightarrow \bar{\varphi}^*$ is a fixed point of T on $B(h,k)$. $\|\bar{\varphi}^*\|_{\text{sup}} = k$ (by definition) $= \|T\bar{\varphi}^*\|_{\text{sup}} \leq \frac{k}{2} \Rightarrow$ contradiction, q.e.d.



Continuity

Definition Let (X, d_x) and (Y, d_y) be metric spaces, and $f: X \rightarrow Y$. let $x \in X$. We say that f is continuous at x if $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $d_x(y, x) < \delta$ then $d_y(f(y), f(x)) < \epsilon$.

Equivalently: $\forall \epsilon > 0 \exists \delta > 0$ s.t. $y \in B^0(x, \delta) \Rightarrow f(y) \in B^0(f(x), \epsilon)$; or $f(B^0(x, \delta)) \subset B^0(f(x), \epsilon)$.

We say that f is continuous if it is continuous at each $x \in X$.

e.g. ① $(\mathbb{R}, |\cdot|)$: we get usual definition of continuity.

② let X be a discrete space, Y be any metric space, f be any function from X to Y . claim: any f is continuous.

let $x \in X$, let $\epsilon > 0$. We want $\delta > 0$ s.t. $f(B^0(x, \delta)) \subset B^0(f(x), \epsilon)$. $B^0(x, \delta) = \{x\}$ if $\delta \leq 1$, X if $\delta > 1$.

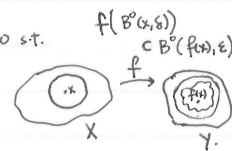
choose $\delta \leq 1$, say $\delta = \frac{1}{2}$. then $f(B^0(x, \frac{1}{2})) = f(\{x\}) = \{f(x)\} \subset B^0(f(x), \epsilon) \Rightarrow f$ is continuous at x . x is arbitrary $\Rightarrow f$ is continuous everywhere.

③ $F: C[0,1] \rightarrow C[0,1]$. let $F(f) = f$ (identity map). F is continuous as a function from $(C[0,1], \|\cdot\|_2)$ to $(C[0,1], \|\cdot\|_2)$?

Yes: choose $\epsilon > 0$. We suppose $\|g - f\|_2 < \delta$. then $\|F(g) - F(f)\|_2 = \|g - f\|_2 = \int_0^1 |g(x) - f(x)| dx < \int_0^1 |g(x) - f(x)| dx$.

Apply Cauchy-Schwarz inequality: $\int_0^1 |g(x) - f(x)| dx \leq \sqrt{\int_0^1 1^2 dx} \sqrt{\int_0^1 |g(x) - f(x)|^2 dx} = \sqrt{\int_0^1 |g(x) - f(x)|^2 dx} = \|g - f\|_2 < \delta < \epsilon$. Take $\delta = \epsilon$, q.e.d.

is F continuous as a function from $(C[0,1], \|\cdot\|_{\text{sup}})$ to $(C[0,1], \|\cdot\|_2)$? Yes. choose $\epsilon > 0$. suppose $\|g - f\|_{\text{sup}} < \delta$. then $\|F(g) - F(f)\|_2 = \|g - f\|_2 = \sqrt{\int_0^1 |g(x) - f(x)|^2 dx} \leq \sqrt{\|g - f\|_{\text{sup}}^2 \int_0^1 1 dx} = \|g - f\|_{\text{sup}} < \delta < \epsilon$.



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Is f continuous as a function from $(C[0,1], \|\cdot\|_1)$ to $(C[0,1], \|\cdot\|_2)$? No! To prove this, we need a theorem:

Theorem 3.9 Heine condition / sequential definition of continuity.

Let $f: X \rightarrow Y$ be a function between metric spaces (X, d_x) and (Y, d_y) . f is continuous at $x \in X \iff$ for any sequence $x_n \rightarrow x$ we have $f(x_n) \rightarrow f(x)$.

Proof (\implies) suppose f is continuous at x . Pick arbitrary sequence $x_n \rightarrow x$. We want to show that $f(x_n) \rightarrow f(x)$. Let $\epsilon > 0$. By continuity, $\exists \delta > 0$ s.t.

$f(B^\circ(x, \delta)) \subset B^\circ(f(x), \epsilon)$. Since $x_n \rightarrow x$, $\exists N \in \mathbb{N} \forall n > N$ s.t. $x_n \in B^\circ(x, \delta)$. Hence, the two statements $\implies f(x_n) \in B^\circ(f(x), \epsilon)$

(\impliedby) suppose RHS is true, but f is not discontinuous at x i.e. $\exists \epsilon > 0 \forall \delta > 0 \exists y \in B^\circ(x, \delta)$ s.t. $f(y) \notin B^\circ(f(x), \epsilon)$.

For any $n \in \mathbb{N}$, choose $\delta = \frac{1}{n} \implies \exists y_n \in B^\circ(x, \frac{1}{n})$ s.t. $f(y_n) \notin B^\circ(f(x), \epsilon)$, \implies contradiction, q.e.d.
 $d_x(y_n, x) < \frac{1}{n} \rightarrow 0$ $d_y(f(y_n), f(x)) \geq \epsilon$
 $y_n \rightarrow x$ $f(y_n) \not\rightarrow f(x)$.

e.g. (cont'd) $F(f) = f$, from $(C[0,1], \|\cdot\|_1)$ to $(C[0,1], \|\cdot\|_2)$. We will show that F is not continuous at the point $f \equiv 0$. It suffices to construct $f_n \rightarrow f$ w.r.t.

$\|\cdot\|_1$, $F(f_n) \not\rightarrow F(f)$: i.e. $\|f_n\|_1 \rightarrow 0$ but $\|f_n\|_2 \not\rightarrow 0$. set $f_n(x) = \begin{cases} a_n & 0 \leq x < \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$. Then we have $\|f_n\|_1 = \int_0^1 |f_n(x)| dx = \frac{a_n}{n}$.

$\|f_n\|_2 = \sqrt{\int_0^1 f_n(x)^2 dx} = \sqrt{a_n^2 \cdot \frac{1}{n}} = \frac{a_n}{\sqrt{n}}$. then if $a_n = \sqrt{n}$, for example, $\|f_n\|_1 \rightarrow 0$ but $\|f_n\|_2 \not\rightarrow 0$.

However, f_n were not discontinuous, so for the correct proof, take $f_n = g_n$ rather than just f_n , where g_n indicates

f_n but with a connecting segment of small order

(\oplus) suppose we have three metric spaces, X, Y, Z . If f, g are continuous everywhere, then $g \circ f$ is continuous everywhere $X \rightarrow Y \rightarrow Z$

This follows naturally from our definitions. Let $x_n \rightarrow x$, then $f(x_n) \rightarrow f(x)$ since f is continuous. Then $g(f(x_n)) \rightarrow g(f(x))$ since g is continuous.

Hence, $(g \circ f)$ is continuous at x .

Definition Let X, Y be sets and $f: X \rightarrow Y$. Let $A \subset Y$. The preimage of A under f is the set $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

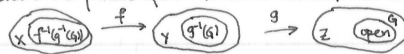
Theorem 3.10 Let (X, d_x) and (Y, d_y) be metric spaces and let $f: X \rightarrow Y$. Then the following three statements are equivalent:

- (i) f is continuous (at every points)
- (ii) for every open set $G \subset Y$ holds $f^{-1}(G)$ is open (the preimage of every open set is open).
- (iii) for every closed set $F \subset Y$ holds $f^{-1}(F)$ is closed (the preimage of every closed set is closed).

comment: this is not true for images! You can have a continuous function f and an open set $A \subset X$ s.t. $f(A)$ is not open.

e.g. $\textcircled{1}$ $f: X \rightarrow Y$, where X is the discrete space and Y is any metric space. This is continuous, since every set in discrete space is open/closed.

$\textcircled{2}$ composition of functions.



If f, g are both continuous, then $g \circ f$ is continuous.

Proof - (i) \implies (ii): let $x \in f^{-1}(G) \implies f(x) \in G$. since G is open, $\exists \epsilon > 0$ s.t. $B^\circ(f(x), \epsilon) \subset G$. since f is continuous (at x),

$\exists \delta > 0$ s.t. $f(B^\circ(x, \delta)) \subset B^\circ(f(x), \epsilon) \subset G \implies B^\circ(x, \delta) \subset f^{-1}(G) \implies f^{-1}(G)$ is open, q.e.d.

(ii) \implies (iii): let $F \subset Y$ be a closed set. Then $Y \setminus F$ is open $\implies f^{-1}(Y \setminus F)$ is open $\implies X \setminus (f^{-1}(F))$ is closed $\implies f^{-1}(F)$ is closed, q.e.d.

(iii) \implies (i): let $x \in X$, let $\epsilon > 0$. $B^\circ(f(x), \epsilon)$ is open since every open ball is open. $\implies Y \setminus B^\circ(f(x), \epsilon)$ is closed.

By assumption, $\implies f^{-1}(Y \setminus B^\circ(f(x), \epsilon))$ is closed and $x \notin f^{-1}(Y \setminus B^\circ(f(x), \epsilon))$ since $f(x) \in B^\circ(f(x), \epsilon)$.

$\implies X \setminus f^{-1}(Y \setminus B^\circ(f(x), \epsilon)) = f^{-1}(B^\circ(f(x), \epsilon))$ is open, and contains x . $\implies \exists \delta > 0$ s.t.

$B^\circ(x, \delta) \subset f^{-1}(B^\circ(f(x), \epsilon)) \implies f(B^\circ(x, \delta)) \subset B^\circ(f(x), \epsilon) \implies f$ is continuous at $x \forall x \in X$, q.e.d.

Compactness.

In \mathbb{R} , Recall: we said that a set of open intervals $\{I_\alpha\}_{\alpha \in A}$ is a cover for a set S if $S \subset \bigcup_{\alpha \in A} I_\alpha$.

The set S is called compact if every cover of S has a finite subcover.

Definition Let (X, d) be a metric space. Let $S \subset X$. A collection of open sets $\{I_\alpha\}_{\alpha \in A}$ is a cover of S if $S \subset \bigcup_{\alpha \in A} I_\alpha$.

A finite subcollection $\{I_{d_1}, \dots, I_{d_m}\}$ is a finite subcover if $S \subset I_{d_1} \cup \dots \cup I_{d_m}$.

The set S is compact if every cover $\{I_\alpha\}_{\alpha \in A}$ of S has a finite subcover $\{I_{d_1}, \dots, I_{d_m}\}$ for some m .

The set S is called sequentially compact if for any sequence $\{x_n\}_{n=1}^\infty$ s.t. $x_n \in S$, \exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ which converges, and $x_{n_k} \rightarrow x \in S$. (its limit is in S).

e.g. $\textcircled{1}$ $(\mathbb{R}, |\cdot|)$ $(a, b) = [a, b] - \text{compact}$ (Heine-Borel theorem), $[a, b] - \text{sequentially compact}$ (Bolzano-Weierstrass theorem).

$(b) S = \mathbb{R}$. Take intervals $\{(-n, n)\}_{n \in \mathbb{N}}$. Then this is a cover, but any finite subcollection does not cover $\mathbb{R} \implies$ no finite subcover \implies not compact.

Take $x_n = n$. this is a sequence which has no convergent subsequence.

(iii) $S = (0,1)$. Consider the intervals $\{(\frac{1}{n}, 1 - \frac{1}{n})\}_{n \in \mathbb{N}}$. This is a cover for $(0,1)$ without a finite subcover. \Rightarrow not compact. For sequential compactness any sequence. By Bolzano-Weierstrass Theorem, \exists a convergent subsequence e.g. $x_n = \frac{1}{n} \rightarrow 0$, however $0 \notin (0,1)$. S is not sequentially compact.

② Let (X,d) be a discrete space. We search for compact sets: let S be a finite set: $S = \{x_1, \dots, x_m\}$. Let $\{I_\alpha\}_{\alpha \in A}$ be a cover of S . \Rightarrow Pick one I_α for each x_i , where $x_i \in I_\alpha$. \Rightarrow finite subcover \Rightarrow every finite set is compact.

For an infinite set, $S = \{x_1, x_2, \dots\}$, we take the cover $\{I_x\}_{x \in S}$. Each set I_x is open. (just the set of singletons). Then $\bigcup_{x \in S} I_x = S \Rightarrow$ this is a cover. However, it has no finite subcover, since every finite subcover would only cover finitely many points. \Rightarrow Every infinite set in the discrete space is not compact.

What about sequentially compact sets? Let S be a finite set, it is sequentially compact, since if $\{x_n\}$ is a sequence in S , it will only take finitely many values \Rightarrow at least one value will be taken infinitely-many times. This is our convergent subsequence.

Now, let S be an infinite set. It is not sequentially compact: Take x_n to be a sequence in X with distinct values. \Rightarrow Each subsequence has distinct values \Rightarrow not eventually constant \Rightarrow does not converge.

Theorem 3.11 A set S is compact $\Leftrightarrow S$ is sequentially compact.

Note: We would only prove the forward direction, the reverse is harder and more time-consuming.

Also, the reverse implication holds in metric spaces, but not in the generalisation to topological spaces.

Proof (\Rightarrow) Assume S is compact, but not sequentially compact. $\Rightarrow \exists \{x_n\}_{n=1}^{\infty}$ s.t. $x_n \in S \forall n$ but x_n does not converge to a point in S .
 $\Rightarrow \forall y \in S \exists r(y) > 0$ s.t. $B^o(y, r(y)) \setminus \{y\}$ does not contain any of the points x_1, x_2, x_3, \dots . Then $\{B^o(y, r(y))\}_{y \in S}$ is a cover for S . Since S is compact, this cover has a finite subcover: $B^o(y_1, r(y_1)), B^o(y_2, r(y_2)), \dots, B^o(y_m, r(y_m))$. The points x_1, x_2, x_3, \dots can only belong to the set $\{y_1, \dots, y_m\}$. At least one of the values will be taken infinitely many times \Rightarrow this produces a constant (hence convergent) subsequence with limit in $S \Rightarrow$ contradicts initial assumption $\Rightarrow S$ is sequentially compact. \square

Definition Let $S \subset X$ be a set in a metric space, (X, d) , $S \neq \emptyset$. The diameter of S is $\text{diam}(S) = \sup_{x,y \in S} d(x,y)$.

e.g. if (X, d) is the discrete space, $S \neq \emptyset$, $S \subset X$, $\text{diam}(S) = \begin{cases} 1 & \text{if } |S| \geq 2 \\ 0 & \text{if } |S| = 1 \end{cases}$.

S is bounded if $\text{diam}(S) < \infty$.

Theorem 3.12 Let (X, d) be a metric space and let $K \subset X$.

- (i) If K is compact then K is bounded (in particular, if K is not bounded, it is not compact).
- (ii) If K is compact then K is closed (in particular, if K is not closed, it is not compact).
- (iii) If K is closed and L is compact, then K is compact.

Proof - (i) Pick $x_0 \in K$. Consider $\{B^o(x_0, n)\}_{n \in \mathbb{N}}$. $\bigcup_{n \in \mathbb{N}} B^o(x_0, n) = X \Rightarrow \{B^o(x_0, n)\}_{n \in \mathbb{N}}$ is a cover for K .

K is compact $\Rightarrow \exists$ finite subcover $\{B^o(x_0, n_1), \dots, B^o(x_0, n_m)\}$. Take $R = \max\{n_1, \dots, n_m\} \Rightarrow K \subset B^o(x_0, R) \Rightarrow \forall x, y \in K, d(x,y) \leq d(x, x_0) + d(x_0, y) < 2R$.
 $\text{diam}(K) \leq 2R < \infty \Rightarrow K$ is bounded, \square .

(ii) Suppose K is not closed $\Rightarrow \exists x \in K, x_n \rightarrow x$ but $x \notin K$. Consider $\{X \setminus B(x, \frac{1}{n})\}_{n \in \mathbb{N}}$, which is open.
 $\bigcup_{n \in \mathbb{N}} X \setminus B(x, \frac{1}{n}) = X \setminus \{x\} \supset K \Rightarrow \{X \setminus B(x, \frac{1}{n})\}_{n \in \mathbb{N}}$ is a cover of K . Since K is compact, there is a finite subcover $X \setminus B(x, \frac{1}{n_1}), \dots, X \setminus B(x, \frac{1}{n_m})$. Let $r = \min\{\frac{1}{n_1}, \dots, \frac{1}{n_m}\}$. $K \subset X \setminus B(x, r) \Rightarrow$ all x_n belong to $X \setminus B(x, r)$.
 $\Rightarrow d(x_n, x) > r$, hence $x_n \not\rightarrow x$, contradiction, \square .

(iii) Let $\{I_\alpha\}_{\alpha \in A}$ be a cover of $K \Rightarrow \{I_\alpha \cap L\}_{\alpha \in A}$ is a cover for L . Since L is compact, there is a finite subcover $I_{\alpha_1}, \dots, I_{\alpha_m}$. $X \setminus K$ for $L \Rightarrow$ these also cover $K \Rightarrow I_{\alpha_1}, \dots, I_{\alpha_m}$ covers K , \square .

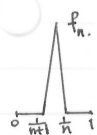
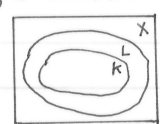
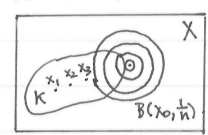
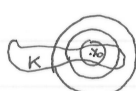
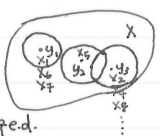
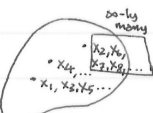
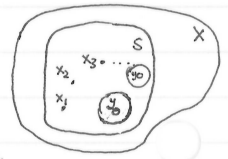
Comments: \mathbb{R}^n with any norm (but not just metric) is compact \Leftrightarrow closed + bounded (reverse direction without proof: do not use in exam).

It is not true in general that compact \Leftrightarrow closed + bounded e.g. (X, d) - discrete space. S is an infinite set \Rightarrow not compact, but S is closed (as every set is closed) and S is bounded, $\text{diam}(S) = 1$.

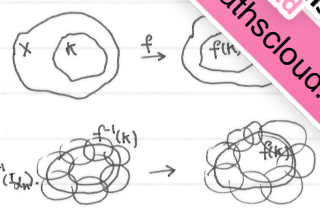
e.g.2. Consider $(C[0,1], \|\cdot\|_{\text{sup}})$. Let $S = B(0, 1)$ be the closed unit ball: S is closed (every closed ball is closed) and bounded ($\|f\|_{\text{sup}} \leq 1 \Rightarrow \|f\|_{\text{sup}} \leq 1$).
 $\Rightarrow \text{diam } S \leq 2$. However, S is not sequentially compact: $\|f_n - f_m\|_{\text{sup}} = 1 \Rightarrow$ no uniform Cauchy subsequence.

\Rightarrow no convergent subsequence. $\Rightarrow S$ is not compact since it is not sequentially compact.

these demonstrate that we require normed spaces!



Theorem 3.13 Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $f: X \rightarrow Y$ be a continuous function. Then, if $K \subset X$ is a compact set, then $f(K) \subset Y$ is a compact set.



Proof - Use definition. Let $\{I_\alpha\}_{\alpha \in A}$ be a cover of A . Consider $\{f^{-1}(I_\alpha)\}_{\alpha \in A}$. This is open, since I_α is open and f is continuous. This is a cover of K . Since K is compact, it has a finite subcover $f^{-1}(I_{\alpha_1}), \dots, f^{-1}(I_{\alpha_n})$. If we map these using f , then $I_{\alpha_1}, \dots, I_{\alpha_n}$ is a finite subcover for $f(K) \Rightarrow f(K)$ is compact. // q.e.d.

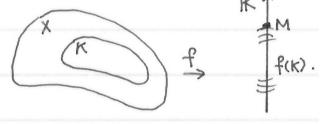
21 March 2013.
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Recall from MATH1101, if $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then f obtains its maximum and minimum. This does not hold for open intervals. This holds, in general, because $[a, b]$ is compact.

Theorem 3.14 Suppose (X, d) is a metric space and $f: X \rightarrow \mathbb{R}$ is continuous. Let $K \subset X$ be a compact set. Then $\exists x \in K$ s.t. $f(x) = \sup_K f$, $\exists y \in K$ s.t. $f(y) = \inf_K f$.

Proof - Since K is compact and f is continuous $\Rightarrow f(K)$ is compact $\Rightarrow f(K)$ is closed and bounded.

Hence $M = \sup f(K) < \infty$. By definition, \sup is least upper bound, $\exists a_n$ s.t. $a_n \in f(K)$, $a_n \rightarrow M$. Since set is closed, $M \in f(K) \Rightarrow \exists x \in K$ s.t. $f(x) = M$. // q.e.d. Second part of proof is analogous, // q.e.d.



Definition Let V be a vector space and $\|\cdot\|, \|\cdot\|'$ be two norms on V . They are called equivalent if $\exists c, C > 0$ s.t. $c\|x\| \leq \|x\|' \leq C\|x\| \forall x \in V$.

Remark - The definition is symmetric w.r.t. the two norms: $\|x\| \leq c\|x\|' \Leftrightarrow \frac{1}{c}\|x\| \leq \|x\|' \leq \frac{1}{c}\|x\|'$.

Comments: This is the same as saying $\forall r > 0 \exists \tilde{r} > 0$ s.t. $B_{\|\cdot\|}(x, \tilde{r}) \subset B_{\|\cdot\|'}(x, r)$: let $r > 0$ be fixed, we choose $\tilde{r} = cr$.

Let $y \in B_{\|\cdot\|}(x, \tilde{r}) \Rightarrow \|y-x\| < \tilde{r} \Rightarrow \|y-x\|' \leq \frac{1}{c}\|y-x\| < \frac{\tilde{r}}{c} = r \Rightarrow y \in B_{\|\cdot\|'}(x, r)$.

The open sets w.r.t. $\|\cdot\|'$ are identical to the open sets w.r.t. $\|\cdot\|$; $\therefore \forall X \in \mathcal{G}$ open, $\exists r$ s.t. $X \in B_{\|\cdot\|'}(x, r)$, but $\exists \tilde{r} > 0$ s.t.

$B_{\|\cdot\|'}(x, \tilde{r}) \subset B_{\|\cdot\|}(x, r)$.

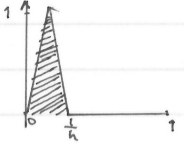
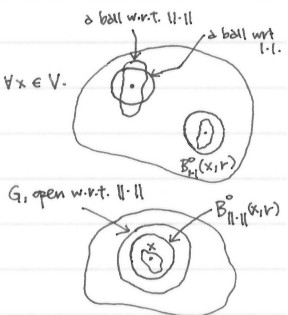
e.g. ① $\mathbb{R}^n: \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|, \|x\|_1 = |x_1| + \dots + |x_n|$. Show that $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are equivalent. $\|x\|_1 \leq n \cdot \|x\|_\infty \Rightarrow \|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty$.

Hence, these norms are equivalent.

② Take $C \in [0, 1]: \|f\|_\infty = \sup_{[0,1]} |f(x)|, \|f\|_1 = \int_0^1 |f(x)| dx$. $\|f\|_1 \leq \int_0^1 \|f\|_\infty dx = \|f\|_\infty$. However, there is no constant c s.t. we have

$\|f\|_\infty \leq c \|f\|_1$. Suppose such a constant exists, define f_n as in the graph, $f_n \in [0, 1]$.

then $\|f_n\|_\infty \leq c \|f_n\|_1 \Rightarrow 1 \leq c \cdot \frac{1}{2n} \rightarrow 0$ as $n \rightarrow \infty \Rightarrow$ contradiction



Theorem 3.15 $\|\cdot\|$ and $\|\cdot\|'$ are equivalent $\Leftrightarrow x_n \rightarrow x$ w.r.t. $\|\cdot\|$ and $\|\cdot\|'$ simultaneously.

Proof - (\Rightarrow) Suppose $x_n \rightarrow x$ w.r.t. $\|\cdot\|$. Then $\|x_n - x\| \leq C \|x_n - x\|' \rightarrow 0$. i.e. $\|x_n - x\|' \rightarrow 0, x_n \rightarrow x$ w.r.t. $\|\cdot\|'$.

(\Leftarrow) Suppose RHS is true but LHS is false, i.e. $\|\cdot\|$ and $\|\cdot\|'$ are not equivalent \Rightarrow one of the inequalities is not true. WLOG, state that $\nexists c$ constant s.t. $\|x\| \leq c\|x\|' \forall x$. Hence, $\forall n \exists x_n$ s.t. $\|x_n\| > n \|x_n\|'$. Define $y_n = \frac{x_n}{\|x_n\|}$. Then $\|y_n\| = \frac{\|x_n\|}{\|x_n\|} = 1 < \frac{1}{n} \rightarrow 0 \Rightarrow y_n \rightarrow 0$ w.r.t. $\|\cdot\|'$.

then $\|y_n\| = \frac{\|x_n\|}{\|x_n\|} = 1 \not\rightarrow 0 \Rightarrow y_n \not\rightarrow 0$ w.r.t. $\|\cdot\|$. This is a contradiction, // q.e.d.

Exercise: Show that $\|\cdot\|, \|\cdot\|'$ are equivalent \Leftrightarrow they have the same collection of open sets.

Theorem 3.16 All norms on \mathbb{R}^n are equivalent.

Comment: This is not true for metrics: e.g. $\mathbb{R}^d, d(x, y) = \begin{cases} 1 & x=y \\ 0 & x \neq y \end{cases}$.

Proof - We will prove that any norm $\|\cdot\|$ is equivalent to $\|\cdot\|_2$. ($\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$). Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$. Let $x = x_1 e_1 + \dots + x_n e_n$.

Then $\|x\| = \|x_1 e_1 + \dots + x_n e_n\| \leq \|x_1 e_1\| + \dots + \|x_n e_n\| \leq |x_1| \|e_1\| + \dots + |x_n| \|e_n\| \leq \sqrt{x_1^2 + \dots + x_n^2} \sqrt{\|e_1\|^2 + \dots + \|e_n\|^2}$ by Cauchy-Schwarz inequality. i.e. $\|x\| \leq C \|x\|_2$ where $C = \sqrt{\|e_1\|^2 + \dots + \|e_n\|^2}$. For the other direction, take $f: \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $f(x) = \|x\|$.

Let $K = \{x \in \mathbb{R}^n: \|x\|_2 = 1\}$. Then K is bounded: $\|x-y\|_2 \leq \|x\|_2 + \|y\|_2 = 1+1=2 \Rightarrow \text{diam}(K) = 2$.

Also, K is closed: $K = B(0, 1) \setminus B^o(0, 1) = \overline{B(0, 1)} \cap (\mathbb{R}^n \setminus B^o(0, 1)) \Rightarrow K$ is closed. \Rightarrow in \mathbb{R}^n , closed + bounded implies compact.

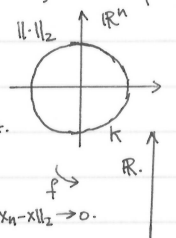
Hence, K is compact. Also, we claim f is continuous: take $x_n \rightarrow x$, NP: $f(x_n) \rightarrow f(x)$. Then $|f(x_n) - f(x)| = |\|x_n\| - \|x\||$.

We use the fact that $\|\cdot\|$ is a norm to apply triangle inequality: $|\|x_n\| - \|x\|| = |\|x_n - x\| - \|x\|| \leq (\|x_n - x\| + \|x\|) = \|x_n - x\| + \|x\| \leq C \|x_n - x\|_2 \rightarrow 0$.

Thus, f is continuous. $\therefore f(x_n) \rightarrow f(x)$. Hence, by Theorem 3.14, f attains its infimum on K , so f is continuous and K is compact.

Denote $c = \inf_K f$. Then $\exists x_0 \in K$ s.t. $c = f(x_0) = \|x_0\| > 0$. Let $x \in \mathbb{R}^n, x \neq 0 \Rightarrow f(\frac{x}{\|x\|_2}) \geq c \Rightarrow \frac{\|x\|}{\|x\|_2} \geq c \Rightarrow \frac{\|x\|}{\|x\|_2} \geq c$.

$\Rightarrow \frac{\|x\|}{\|x\|_2} \geq c \Rightarrow \|x\| \geq c \|x\|_2$. Hence, $c \|x\|_2 \leq \|x\| \leq C \|x\|_2 \Rightarrow \|\cdot\|$ and $\|\cdot\|_2$ are equivalent, // q.e.d.



Some words on the exam:

Non-examinable content: equivalent norms, Picard's theorem.

6 questions: 2 on uniform convergence, 1 on Fourier series, 3 on metric spaces. Similar to previous years.

Each question involves bookwork + problems to solve. Focus on homework and examples discussed in class.